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THE EFFECT OF AUCTION FORMAT ON EFFICIENCY AND REVENUE IN DIVISIBLE GOODS AUCTIONS: A TEST USING KOREAN TREASURY AUCTIONS*

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This paper measures the efficiency and revenue properties of the two most popular formats for divisible goods auctions: the uniform-price and discriminatory auction. We analyze bids into the Korean Treasury auctions which have used both formats. We find that the discriminatory auction yields statistically higher revenue. Unlike previous work that uses data from only one format, we are able to compare the efficiency properties of the two formats. We find that the discriminatory auction better allocates treasury bills to the highest value financial institutions. However, the differences in revenue and efficiency are not large because the auctions are very competitive.

I. INTRODUCTION

Divisible goods auctions are becoming an increasingly popular means to transact in some markets. Examples include markets to purchase treasury bills and environmental emission permits, spot markets to sell electricity, and initial public offerings of stock. Theory does not predict the auction format that will perform better in terms of raising revenue or efficiently allocating the goods. Divisible goods (or multiunit) auctions differ from the more traditional auctions for a single object, for which rich theory has been developed. In multiunit auctions, bidders submit multiple price-quantity pairs to form demand or supply schedules to buy or sell multiple units of the good. For example, in a sealed-bid auction to purchase a fixed quantity of goods, all bidders' bid schedules are summed to form an aggregate demand schedule. The intersection of aggregate demand and total supply determines the market-clearing price, and all bids above the market-clearing price win the right to purchase the goods. The rule determining the price paid by each winning bidder has important effects on the efficiency of the auction. Several auction formats have been proposed including the uniform price (or single-

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price auction), the discriminatory (or 'pay-your-bid' auction), and the Vickrey auction. In uniform-price auctions, the winning bidders pay the market-clearing price for all units purchased. In discriminatory auctions, winning bidders pay the price bid for each unit. And in Vickrey auctions, bidders winning k units pay the sum of the k highest losing bids. The two most common formats used in practice are uniform-price and discriminatory auctions. For example, most electricity auctions use the uniform-price format. The U.S. Environmental Protection Agency sells SO2 permits using a pay-your-bid auction. Treasury auctions for government securities around the world have used both formats. Despite the widespread use of both formats, the efficiency and revenue properties of each format in multiunit settings are theoretically ambiguous.[1]

An incomplete understanding of the bid shading incentives under each format has led to a lively debate about the optimal auction format. The efficiency and revenue from each format is determined by the relative incentives to shade bids below the marginal valuation of the goods. Early work relied on intuition from theoretical results about first and second-price auctions for a *single* unit. However, such intuition does not generalize to bidding strategies in multiunit auctions.[2] In multiunit auctions using either the uniform-price or discriminatory pricing rule, bidders have incentives to shade their bids below the valuation and the incentive may differ for each unit demanded. Suppose a bidder submits a demand schedule with a different price for each of x units of a good. In uniform-price auctions, the bidder has an incentive to bid below valuation for the x^{th} unit of a good if that bid has some probability of lowering the market-clearing price that is paid for all other x-1 units. This result is very similar to a monopsony buyer who reduces demand in order to drive down the market price. The incentive to bid shade is different in discriminatory auctions. Bidding the marginal valuation would yield no gain in a pay-your-bid auction because the payoff is zero in the event of winning. Therefore, bidders have incentives to shade bids on all units with the amount of shading determined by the bidder's belief about the location of the market-clearing price.[3]

[1] Comparisons of the efficiency and revenue for various auction formats have been established for *single* unit auctions. Vickrey [1961] demonstrates the well-known Revenue Equivalence Theorem that if risk neutral bidders adopt non-cooperative strategies in the private independent value paradigm, then the expected selling price is the same for first and second price sealed-bid, English and Dutch auctions. Comparisons also have been established for more general value structures. For the risk-averse bidders, Holt [1980] shows that the seller strictly prefers the Dutch or first-price auction to the English or second-price. Milgrom and Weber [1982] show that if risk neutral bidders have affiliated values, the four common single auction mechanisms can be ranked as the English auction, the second-price auction, and the Dutch which yields the same revenue as the first-price auction.

[2] For a very clear introduction to multi-unit auctions, see Krishna [2002], part II.

[3] The equilibrium bidding incentives are analyzed in a variety of theoretical papers including Ausubel and Cramton [2002], Engelbrecht-Wiggans and Kahn [1998a, b], Swinkels [1999, Wang and Zender [1996], Noussair [1995], Back and Zender [1993, and Wilson [1979].

The choice of auction format has been debated in both the electricity and emission permit markets, but the market design has been most controversial in treasury auctions.[4] Because an important goal is to maximize revenue to the government, the most common criterion to evaluate auction format is revenue. The format has changed from discriminatory to uniform-price in many countries such as the United States, Sweden, Switzerland, Mexico, Turkey and Korea. The well-known Revenue Equivalence Theorem for an independent private values setting cannot be applied because the bid shading incentives imply that the two formats do not lead to the same allocation – a necessary condition for revenue equivalence. Thus, the issue of maximum revenue is treated as an empirical question.

Both reduced-form and structural empirical analyses have addressed the question of which format maximizes revenue, and the findings are generally mixed. Recent structural analyses have developed empirical methodologies that use bid data to estimate bidder valuations and conduct counterfactual experiments. Hortaçsu [2002] analyzes bids into Turkey's Treasury bill auction which used a discriminatory format. Hortaçsu derives a structural model of bidding that maps bidder valuation to equilibrium bids. By inverting the bid function, he uses observed bids to recover the underlying valuations by each bidder for each unit of T-bills. He uses the estimated valuations to construct bounds on the revenue if those bidders had instead participated in a uniform-price auction. He finds that the discriminatory auction yields more revenue *ex-post*, but he cannot reject equivalence in *ex-ante* expected revenue. Most recently, Kastl [2006] studies bidding into the Czech government's treasury auction that utilized a uniform-price format. Methodologically, Kastl characterizes equilibria in step functions and analyzes the implications of econometric methods that assume continuously differentiable versus step bid functions, as we discuss in more detail below. He finds that the uniform-price treasury auctions used by the Czech government yield allocations that are nearly efficient and failed to extract less than 3 basis points of bidder surplus. Other papers that make different assumptions about the value structure have analyzed the French and Mexican Treasury auctions (Fevrier *et al.* [2004] and Castellanos and Oviedo [2005]).

Efficiency is an important criterion that has received less attention in empirical analyses. In the auction for any good with a private value component, market designers should be concerned whether the auction format allocates the goods to the buyers with the highest valuation. There is no general ranking of the efficiency of either format in a multiunit setting. The differential incentives to shade bids for different units of the good imply that inefficient equilibria are obtained in both the uniform-price (Engelbrecht-Wiggans and Kahn [1998b]) and discriminatory auctions

[4] For a survey, see Bikchandani and Huang [1993].

(Engelbrecht-Wiggans and Kahn [1998a] and Swinkels [1999]). Intuitively, bid shading that varies across units demanded or bidders implies that the order of bid prices does not correspond to the order of valuations. In uniform-price auctions, bidders have increasing incentives to shade bids for larger demand quantities, so efficiency is unlikely unless, for example, each bidder proportionately shades in the same way. For discriminatory auctions, the incentive to shade does not necessarily increase in quantity demanded, but there are strong incentives to significantly mark down bids for units with values significantly higher than the expected market-clearing price. In general, the performance of uniform-price and discriminatory auctions depends upon the distribution of bidders' valuations, and either format potentially can be more efficient (Ausubel and Cramton [2002]). In order to study the relative efficiency of uniform versus discriminatory pricing, the researcher must have data on actual bidding under both formats, as we have for the Korean Treasury auctions.

This paper makes two contributions to the existing empirical literature on multiunit auctions. First, we follow the approach of Hortaçsu [2002] who uses a structural model of equilibrium bidding to analyze discriminatory auctions. We extend this methodology to uniform-price auctions. We non-parametrically recover estimates of the bidders' valuations, and use the valuations to conduct counterfactual revenue comparisons. Because we have bid data from two auction formats, we are able to compare the revenue from discriminatory and uniform-price auctions to a common benchmark. We find that both formats in Korea are very competitive. Although revenue in the discriminatory auction is statistically larger than in the uniform-price auction, the monetary difference is not large.

Our second and more significant contribution is to analyze the efficiency properties of each format. Both formats are inefficient mechanisms, and there is no general theoretical ranking of their efficiency. As a result, it is not possible to evaluate the relative efficiency of each format without using bid data generated by both formats. Existing structural analyses of bidding have been unable to study the relative efficiency of each format because researchers have not utilized bidding data from a country that switched from one format to the other. To our knowledge, this is the first analysis of field data to assess the relative efficiency of the two formats for multiunit auctions. We find the discriminatory auction to be more efficient but the difference is not large due to the competitiveness of the Korean market.

This paper is organized as follows: in section II, we discuss the institutions for trading government securities in Korea. Section III describes structural models of equilibrium bidding into both discriminatory and uniform-price auctions. The models allow us to use individual-level bid data to recover estimates of bidders' valuations. Section IV contains the empirical results and comparisons of the two auction formats in terms of revenue and efficiency. Section V concludes.

II. THE KOREAN TREASURY AUCTION MARKET AND DATA

The Korean Treasury has auctioned government securities using standard auctions since 1999. Initially, the discriminatory pricing rule was utilized, but the Treasury switched to uniform pricing in August, 2000. Securities of varying maturities are sold, but we use the 3 year bond which comprises the largest volume. The 3 year bond is auctioned on the second Monday of every month and the quantity sold is pre-announced about 3-5 days prior to sale. We use data for September, 1999, to April, 2002, which include 10 discriminatory and 20 uniform-price auctions.[5]

The bidders are restricted to financial institutions that are designated as Primary Dealers (PDs). The number of PDs varies between 27 and 30 during our sample period and are about evenly divided between commercial banks or security houses (or brokerage firms). Long-term investors such as pension funds, investment trust companies, and insurance companies cannot be designated as PDs, so they must submit their bids through a PD. PDs are required to meet specific financial requirements and were reevaluated twice during our sample in September, 2000, and December, 2001. The number of PDs was: 30 (16 banks and 14 brokerages) from September, 1999–September, 2000; 30 (14 banks and 16 brokerages) from October, 2000–December, 2001; and 27 (11 banks and 16 brokerages) from January, 2002, until the end of our sample. The identities of the registered PDs were public information, so the number of potential bidders in each auction can be characterized as common knowledge. Most PDs participated in each auction. In the discriminatory auctions, the mean number of bidders was 27.9 and the standard deviation was 2.3; in the uniform-price auctions, the mean and standard deviation of the number of bidders were 24.9 and 1.35, respectively. Table I shows that the fluctuation in the number of bidders across time is relatively small. In addition, there were four particular PDs who frequently did not bid or purchased relatively small quantities (two banks and two brokerages). Among these frequent non-participants, two merged with other PDs in the middle of the sample period and the other two later failed to be redesignated as PDs. This suggests that from the perspective of an individual PD, the number of other bidders was not known with certainty but did not vary substantially from auction to auction.

Bid schedules are submitted by 3:00pm on Monday and auction results are announced by 4:30pm. Bid schedules specify a yield and quantity demanded at that yield. The maximum number of yield-quantity pairs is five. The bid schedule is constructed as a step function connecting each of the yield-quantity pairs. Quantities are specified in terms of the face value of securities with a minimum bid increment of 0.1 billion KW (about $83,000). The bid yield also has a minimum increment of 0.01 percentage points which

[5] We drop two auctions due to lack of appropriate data.

TABLE I
SUMMARY STATISTICS ON BIDDING DATA AT EACH AUCTION

			Bid amount (Bill KW)			Winning amount (Bill KW)					Bid price (KW)		
Format	Number of Bidders	Date	Mean	Max	Min	Mean	Max	Min	Avg. # of Bid Points	Avg. winning Bid Points	Mean*	Max	Min
D	28	9/13/99	90.5	300	10	57.0	300	5	3.82	2.00	9996.74	10031.09	9943.29
D	30	10/11/99	115.0	350	10	52.2	203.1	7.2	3.67	1.80	9996.29	10013.15	9968.53
D	28	11/15/99	79.9	290	10	47.4	200.3	8.6	3.86	2.00	9998.74	10013.15	9971.14
D	29	1/17/00	78.4	180	20	28.3	70.7	2.1	3.69	1.83	9996.10	10015.49	9974.24
D	30	2/14/00	141.0	385	20	45.5	145	5	3.83	1.93	9998.75	10013.03	9989.59
D	29	3/13/00	65.9	150	10	26.6	115	2	3.34	1.76	9997.71	10010.41	9974.03
D	30	4/10/00	101.3	340	20	36.5	210	2	3.47	1.17	9997.30	10010.45	9984.35
D	27	5/8/00	71.1	165	10	34.6	72	1	3.19	1.56	9998.41	10007.83	9986.96
D	23	6/12/00	42.4	180	10	28.9	108.7	1	2.91	2.17	10002.09	10023.59	9973.86
D	25	7/10/00	45.6	120	10	27.9	100	1	2.52	1.64	9999.51	10018.54	9981.50
UP	23	8/14/00	58.7	130	15	29.1	81	5	3.22	1.74	9999.45	10069.33	9973.48
UP	24	9/18/00	52.5	170	10	42.9	156	7	2.38	1.67	10000.96	10029.06	9908.17
UP	23	10/9/00	84.1	220	10	40.7	184	5	3.48	1.91	9999.80	10018.57	9984.11
UP	25	11/13/00	92.4	210	10	45.2	137	9	3.28	1.52	9996.65	10021.51	9943.80
UP	26	1/8/01	72.9	220	10	41.4	100	10	2.85	1.62	9999.24	10030.05	9975.49
UP	24	2/5/01	48.8	160	10	36.7	100	10	2.00	1.25	10008.31	10055.23	9969.77
UP	23	3/12/01	33.5	80	10	21.7	60	5	1.70	1.39	10007.00	10057.36	9945.72
UP	26	4/2/01	65.4	240	10	38.1	160	10	2.46	1.58	9993.69	10070.52	9903.28
UP	25	5/7/01	54.0	150	10	28.6	110	5	2.36	1.28	9999.42	10048.82	9946.07
UP	26	6/4/01	56.2	150	10	21.1	50	5	2.69	1.08	9996.59	10030.00	9972.82
UP	27	7/2/01	67.8	140	10	33.3	90	10	3.26	0.70	9987.33	10019.15	9950.95
UP	26	8/6/01	76.9	220	10	38.9	195	5	3.04	1.31	9997.25	10024.82	9967.02
UP	24	9/3/01	74.6	180	10	40.5	180	10	2.88	1.71	10000.04	10038.84	9966.84
UP	26	10/8/01	92.7	180	20	35.6	90	10	3.65	1.77	9993.72	10030.81	9958.16
UP	26	11/7/01	61.2	140	10	30.0	80	10	2.81	1.77	9997.75	10038.94	9966.76
UP	25	12/3/01	73.2	180	10	45.8	115	10	3.24	2.24	10002.01	10032.96	9986.30
UP	27	1/7/02	161.5	500	30	57.1	240	5	4.37	1.63	9994.36	10032.73	9953.84
UP	24	2/4/02	59.6	150	10	28.6	80	5	2.71	1.08	9995.83	10027.34	9970.03
UP	23	3/4/02	75.7	150	10	29.4	85	5	3.35	1.30	9994.93	10027.35	9972.74
UP	24	4/1/02	73.3	140	10	35.6	80	10	3.21	0.88	9993.60	10027.12	9970.26

Notes: D = Discriminatory, UP = Uniform-Price. The distribution of the PDs between banks and brokerages is: September 1999-September 2000: 16 banks and 14 brokerages; October 2000-December 2001: 14 banks and 16 brokerages; and January 2002 until the end of the sample: 11 banks and 16 brokerages.

*quantity weight average bid price = $\sum_{i=1}^{N}\sum_{k=1}^{K}(b_{ikt} \times q_{ikt}) / \sum_{i=1}^{N}\sum_{k=1}^{K} q_{ikt}$ where (b_{ikt}, q_{ikt}): bidder i's k^{th} bid price and bid amount (increment) at auction t

represents an increase or decrease by about 3KW. Yields can be converted to prices as the amount a bidder is willing to pay for one unit of a treasury bill with a face value of 10,000 Korean Won (KW) and a certain coupon accruing periodic interest payments.[6]

Our data consist of the quantity of treasury bills sold, each bidder's set of yield-quantity pairs that comprise the bid schedule, and the market clearing yield for each auction. Because bids are in yield, we convert yield to prices and normalize by setting the market clearing yield for each auction to be 10,000 KW. Quantities are also normalized by the ratio to the total supply of each auction so that total supply of each auction is normalized to be 1. Summary statistics for each auction are shown in Table I. Bidders use less than four bidpoints on average, despite the fact that five yield-quantity points are permitted. This suggests that the limit on the number of allowed bidpoints is not a binding constraint for many of the PDs. Table I suggests that bid schedules are flatter in discriminatory auctions – the difference between the minimum and maximum bid price is smaller on average during the discriminatory period.

III. STRUCTURAL MODEL OF BIDDING AND EMPIRICAL STRATEGY

The existing empirical literature on multiunit auctions typically employs one of two empirical strategies.[7] One approach uses changes in auction format in a particular market as a 'natural experiment'. Researchers measure the difference between the true value and bid price as the differential between the when-issued (or resale) price and the auction price.[8] This approach tests if the differential is larger, controlling for observables, for one of the pricing rules using data before and after a change in format. Umlauf [1993] finds the uniform-price auction yields higher revenue using data from Mexican 30-day Treasury Bill auctions. Simon [1994], Nyborg and Sundaresan [1996] and Malvey and Archibald [1998] analyze U.S. data. Simon finds that uniform-price auctions yield less revenue while the latter two do not find statistically significant differences. This approach is appealing because it is based on the experiment one would like to conduct, but it relies on the

[6] To convert bid yield to bid price, we use the same convention that is widely used in the market:

$$price = \sum_{i=1}^{11} \frac{it}{\left(1+\frac{yield}{4}\right)^{i}} + \frac{10000+it}{\left(1+\frac{yield}{4}\right)^{12}}$$

where it = quarterly interest payment accrued by the coupon rate. The coupon rate is set to the quantity-weighted average of the winning yield after the auction except for several cases during the uniform-price period when it was announced by the Treasury before the auction.

[7] For a general survey on empirical analysis of auctions, see Hendricks and Paarsch [1995].

[8] When-issued trading occurs during the period between the auction announcement date and the actual issue date of the security. Prior to the Treasury's scheduled auction date, dealers and investors may take either long positions or short positions in the security to be auctioned.

assumption that bidders' true valuations are accurately reflected in resale markets, and that one can control for any changes in information from the close of the auction until aftermarket trading.[9]

The second approach uses a structural model of bidding to map observed bids into estimates of the unobserved valuations. Given an assumed value structure, one can construct a strategic model of equilibrium bidding under either of the two pricing rules. If the equilibrium bid function is invertible, the observed bids can be used to calculate the underlying valuation for the goods. Under the assumptions that the value structure (e.g., independent private values) is correctly specified and that bidders are playing a strategic equilibrium, the researcher can nonparametrically identify the valuation function. Equipped with the estimated demand function, the researcher calculates the revenue under other strategic models of bidding. For example, one can calculate revenue under the Vickrey auction in which it is a dominant strategy equilibrium to bid the true value function. Hortaçsu [2002] uses such a methodology to compare revenue in Turkey's discriminatory Treasury auctions to counterfactual revenue under a uniform-price auction.[10]

We extend this structural methodology to the Korean auctions which use both formats. In addition, we use our estimated valuations to calculate the efficiency of allocations under each format. We characterize an equilibrium model of bidding into both the discriminatory and uniform-price auctions. The model allows us to derive a first-order condition that maps valuation functions to bid schedules, and we use the condition to estimate the underlying valuation function for each bidder. The estimated 'demand function' allows us to characterize revenue and efficiency under counterfactual auction formats. We compare the welfare under the actual auction format to a benchmark for an efficient auction.

We model bidding as a static game in which bidders maximize expected profits.[11] Each bidder is assumed to be *ex-ante* symmetric and have independent private values for the treasury bills. At the time when bids are submitted, the PDs are assumed to know the total supply of securities for sale, the total number of bidders, and their own value function. Bidders do

[9] Other papers that investigate the incentive to shade bids include Smith [1966], Harris and Raviv [1981], Scott and Wolf [1979, and Tenorio [1997]. Papers that investigate bidding strategies include Commack [1991], Wolfram [1998], Hortaçsu [2001], and Nyborg, Rydqvist and Sundaresan [2002].

[10] Heller and Lengwiler [1998] use a similar methodology to analyze the Swiss Treasury market.

[11] By choosing a static model, we do not allow for collusion or some form of dynamic pricing. Umlauf [1993] tests the change in the degree of the collusion in the Mexican Treasury market using the dispersion of bid prices as the proxy of the degree of competition. However, unlike in the Mexican market where the six largest bidders purchase over 70% of total supply, the Korean market is far less concentrated. Also, conversations with some primary dealers and researchers who specialize in the Korean market suggest that collusion is not likely.

not know their rivals' valuations but have a common prior on the distribution of the valuation function.

These assumptions appear reasonable for the bidders in these auctions. The PDs can be modeled as risk neutral because treasury bills comprise a small fraction of total assets and there are various other low risk financial instruments in which to invest. Of course, if treasury bills are the sole source of fulfilling a reserve requirement, then bidders face the risk that losing in the auction forces them to purchase from the secondary market. The assumption that bidders are *ex-ante* symmetric implies that each bidder has the same distribution of latent demand. This may be questionable because PDs represent both banks and security houses with different underlying motivations to purchase government securities. In addition, primary dealers appear to differ in size. Nevertheless, we assume symmetry for the benchmark model. In order to address possible asymmetries, we repeat the analysis below where we assume bidders to be symmetric within group and allow for two different groups of bidders with different distributions of the private signal. Allowing for asymmetries does not change our conclusions, and the results testing for robustness to asymmetries are shown in section IV(iv).

The most important assumption is the underlying value structure. Typically, researchers assume either a private value setting (in which the bidder knows her own value but not the rivals' values), a common value setting (in which the good has a unique common value but bidders have different signals of the common value) or a more general affiliated value setting. The value structure could be modeled as common value if each bidder's motivation for purchasing treasury bills is to trade them in secondary markets where there is one common future price but traders have different forecasts of that price.[12] However, the value structure is better characterized as private values if winning bidders hold treasury bills up to maturity. We cannot rule out a more general value structure because bidders can participate in the auctions for both purposes.

Unfortunately, the existing literature does not provide robust tests for the underlying value structure in a multiunit auction. There exist three possible sources of private information in this market.[13] First, each bidder may have a different reserve requirement for treasury bills or a different availability of liquidity which is not known to their rivals. Second, primary dealers serve as intermediaries to purchase securities for other firms, and each primary dealer may have a different level of commitment to place orders for customers. The terms of the purchase arrangements can make winning units in the auction more valuable to some PDs than to others. Finally, each

[12] Fevrier, Preget and Visser [2004] and Castellanos and Oviedo [2005] make a common value assumption in their analyses of the French and Mexican markets.
[13] Hortaçsu [2002] identifies the first and third as possible sources of private information.

bidder may have different forecasts of long-run interest rates which generate different values to holding government securities. Because of these sources of bidder-specific values, we assume a private value model as several other papers in the empirical literature have done. Below we argue that any bias introduced by assuming private rather than common values is unlikely to affect our revenue ranking. We assume each bidder's 'demand function' is decreasing in price so marginal value per unit is weakly decreasing in quantity.

III(i). *Model*

We model strategic bidding in these divisible goods auctions as an application of Wilson's [1979] share auction model. The share auction model is most easily understood when the bid schedules are modeled as smooth, continuous functions. However, the bid schedules in the Korean Treasury auctions are discrete 'step' functions. Therefore, we model bidding in a discrete strategy space in which firms submit a finite number of bidpoints that are connected with a step function. This formulation follows techniques in Nautz [1995] and Hortaçsu [2002] in which perfect divisibility of the quantities is maintained but restricted to lie on a discrete price grid. After developing the discrete version of the model, we show the continuous version analog which has a simple intuitive interpretation.

Let the total supply be Q and the number of bidders be N (denoted by $i = 1,\ldots,N, N \geqslant 2$) which is assumed to be commonly known to each bidder. As we discuss in section II, the identities of potential bidders (i.e., PDs) are public information and the number of actual bidders is relatively stable over time. (In section IV(v), we modify our model to allow for uncertainty in the number of actual bidders, but the empirical results are very similar). Bidders are assumed to be risk neutral. Let $v_i(\cdot)$ be the true marginal valuation (or demand) function for Treasury bills of bidder i, t_i be the private signal only known to i, and s be a commonly known signal among bidders. The general bidder i marginal valuation function is given by $v_i = v_i(q, t_i, s)$ with $v_q \leqslant 0$ and t_i and s possibly correlated. However, because we restrict valuations to be independent private values, the valuation function is given by: $v_i(q, t_i, s) = v(q, t_i)$.

We define an arbitrarily fine grid of prices given by a vector p with elements separated by Δp: $p_0 < p_1 < \cdots < p_{K+1}$

The bid vector submitted by bidder is a series of quantities specified for each of these prices: $\vec{y}_i : \{y_{io} \geqslant y_{i1} \geqslant \cdots \geqslant y_{iK+1}\}$

Even if a bidder only submits five price-quantity pairs, the step function formed by those bidpoints implicitly defines bid quantities for all points on the price grid.

After all bids are submitted, the Treasury determines the market-clearing price by aggregating the quantity bids for each point on the price grid and finding the price at which the total demand falls just short of the total supply:

$$p_{k^*} : k^* = \min\{k : \sum_{i=1}^{N} y_{ik} \leqslant Q\}^{14}$$

At the market-clearing price, p_{k^*}

$$y_i(p_{k^*}) \cong Q - \sum_{j \neq i}^{N} y_j(p_{k^*})$$

In words, $p_{k^*}$is the price at which the bid schedule from bidder i intersects residual supply, where residual supply is the aggregate rival bid schedule subtracted from the total quantity supplied. Define the distribution function of the market-clearing price conditional on submitting the bid vector of bidder $i, \vec{y}_i$ as:

$$H(p_k, \vec{y}_i) \equiv \Pr\left\{ y_{ik} \leq Q - \sum_{j \neq i}^{N} y_{jk} \right\} = \Pr\{p_{k^*} \leq p_k | \vec{y}_i\}$$

Intuitively, this says that conditional on bidder i submitting a particular bid function $\vec{y}_i$, the probability that there is excess supply at any price p_k is given by $H(\cdot)$. Because excess supply at price p_k implies that the market-clearing price is lower than p_k, $H(\cdot)$ also defines the probability that the market-clearing price $p_{k^*}$is less than p_k. $H(\cdot)$ allows us to collapse all uncertainty faced by bidder i into a single function. In particular, private information possessed by rival bidders determines the rivals' bid schedules and, hence, the market-clearing price. Because bidder i does not know rivals' private information when submitting her bid, the bidder faces uncertainty regarding the equilibrium market clearing price. The distribution of market-clearing price is assumed to be continuous and differentiable with respect to the quantity. Below we use data from the auctions to estimate $H(\cdot)$. We model equilibrium behavior under two auction formats, so let the conditional distribution of the market-clearing price be given by $H(\cdot)$ for discriminatory auctions and $G(\cdot)$ for uniform-price auctions.

Discriminatory Auction In the discriminatory auction, the expected payoff of a risk neutral bidder who submits the bid vector, $\vec{y}_i$ is

[14] This definition of the market-clearing price avoids the complication posed by rationing. Rationing occurs when the total supply intersects aggregate demand at a flat spot in the aggregate demand step function – more units are demanded than are available at the market-clearing price. Participants involved in the Korean auctions claim that rationing is not a major factor. Nevertheless, when we compare the revenue or surplus below, we apply the actual rationing rule used by the Treasury.

given by:[15]

$$\sum_{k=0}^{K+1} [\Pr\{Mkt\ Clr\ P = p_k; \vec{y}\}] * \{payoff\ on\ bids \geqslant p_k\}$$
$$= \sum_{k=0}^{K+1} [H(p_k, \vec{y}) - H(p_{k-1}, \vec{y})] \times \sum_{j=k}^{K+1} \left(\int_{y_{j+1}}^{y_j} v(q, t_i) dq - p_j(y_j - y_{j+1}) \right)$$

The bidder maximizes expected profits by choosing price-quantity bid pairs subject to two constraints: (1) a maximum of five bid pairs, and (2) the bid schedule is monotonic. Because many bidders do not fully utilize the five available bidpoints, we exclude the number of bidpoints restriction.[16] The monotonicity restriction is explicitly incorporated into the formulation. In the main text, we show the first-order necessary conditions and provide intuition for their interpretation. The derivations can be found in the Appendix.

If the bidder submits strictly increasing quantity bids at every point on the price grid, then the monotonicity constraints are not binding, $\lambda_k = 0 \forall k$. The first-order condition expresses valuation as function of price and a 'bid shading' term:

$$v(y_k, t_i) = p_k + \frac{H(p_{k-1})[p_k - p_{k-1}]}{H(p_k) - H(p_{k-1})} - \frac{\frac{\partial H(p_k)}{\partial y_k} \left(\int_{y_{k+1}}^{y_k} v(q, t_i) dq - p_k(y_k - y_{k+1}) \right)}{H(p_k) - H(p_{k-1})}$$

We could use this first-order condition to estimate the valuation function if bidders submitted unique quantity bids at every price on the price grid. However, in our data the bidders submit unique quantity bids at only a subset of the possible prices. If a bidder does not submit a unique quantity at p_k, then implicitly $y_k = y_{k+1}$.

In the discriminatory auction, there are several possible interpretations of the fact that a bidder does not submit a bid at every possible price point. One possibility is that there is some (unmodeled) cost to adding a bidpoint, and that cost outweighs the expected benefits of 'fine tuning' the bid function. Intuitively, adding an additional bidpoint at some locations may yield only a small amount of additional expected profit, and a bidder may not find it worthwhile to compute and submit that bidpoint. Kastl [2006] derives a model of bidding in

[15] Because bidders are assumed to be symmetric, the derivation suppresses the i subscript.

[16] Kastl [2006] interprets utilizing less than the maximum allowable bidpoints as there being a cost to submitting refined bid functions, and he describes a method to estimate the implicit cost of 'finetuning' bid functions. Market analysts who were involved in the design of the Korean market say that market participants were asked about the maximum number of bidpoints that should be allowed; most participants suggested that 4-5 points would be sufficient to create a flexible bid function and that allowing more bidpoints was not useful.

step functions that explicitly incorporates the cost of submitting bidpoints, and he estimates bounds on these costs using bid data in the Czech treasury auctions.

Another interpretation, suggested by Nautz [1995] and Hortaçsu [2002], is that the monotonicity constraint is binding at the unobserved bidpoints. In the discriminatory auction, a sufficient condition for the monotonicity constraint to bind is that the distribution of the market-clearing price is flat across the unobserved bidpoints (i.e., $H(p_j) = H(p_{j-1})$ at unobserved bidpoint j). For example, suppose a bidder submits bids at p_2 and p_5 but not at p_3 and p_4. Intuitively, there is no x_3 satisfying $x_5 < x_3 < x_2$ that would change the market-clearing price, and thus the probability of 'winning', under any possible realization of residual supply. This is shown in detail in the Appendix. In our model of the discriminatory auction, unobserved bidpoints (or 'zero bids') are consistent with either interpretation. However, this is not the case for the uniform-price auction in which we must impose more structure on the data, as we show below.

As shown in the Appendix, we can express the valuation at each observed bidpoint in terms of only the observed bidpoints. Letting k^m index the observed bidpoints, the first-order condition can be written as:

$$(1)\qquad v(y_{k^m}, t_i) = p_{k^m} + \frac{H(p_{k^{m-1}})[p_{k^m} - p_{k^{m-1}}]}{H(p_{k^m}) - H(p_{k^{m-1}})} - \frac{\frac{\partial H(p_{k^m})}{\partial y_{k^m}}\left(\int_{y_{k^{m+1}}}^{y_{k^m}} v(q,t_i)dq - p_{k^m}(y_{k^m} - y_{k^{m+1}})\right)}{H(p_{k^m}) - H(p_{k^{m-1}})}$$

In order to calculate the integral involving $v(q,t_i)$, we need to know the functional form of the valuation function between bid quantities. We assume the marginal valuation function is a step function which assumes constant values of $v(y_k^m)$ on (y_k^{m+1}, y_k^m), as illustrated in Figure 1.[17] The integral in the above equation becomes $v(y_{k^m})(y_{k^m} - y_{k^{m+1}})$.Thus, we have derived a set of linear equations that we can solve to estimate the marginal valuation step function.

Uniform-Price Auction In the uniform-price auction where the bid vector is denoted, $\vec{x}_i$, the expected payoff is given by:

$$\sum_{k=0}^{K+1}[\Pr\{Mkt\ Clr\ P = p_k) * (\text{payoff if } Mkt\ Clr\ P = p_k)]$$
$$= \sum_{k=0}^{K+1}[G(p_k,\vec{x}) - G(p_{k-1},\vec{x})]\left(\int_0^{x_k} v(q,t_i)dq - p_k x_k\right)$$

[17] McAdams [2006] shows that under weaker assumptions of non-negative marginal values and/or non-increasing marginal values, one can compute bounds on marginal valuations. Also, Kastl [2006] describes necessary conditions and techniques for non-parametrically identifying the full marginal valuation function.

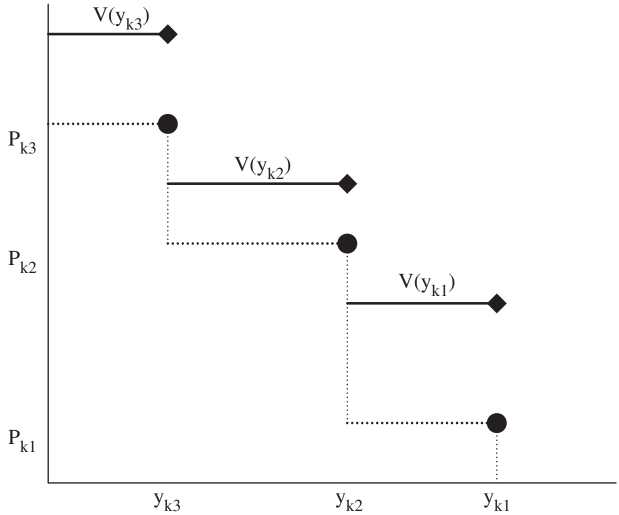

Figure 1
Illustration of Step Valuation Function

As with the other format, if the bidder submits strictly increasing quantity bids at every point on the price grid, then the monotonicity constraints are not binding, $\lambda_k = 0 \forall k$. The first-order condition can be written so that valuation at each bidpoint is a function of the price plus a 'bid shading' term:

$$v(x_k, t_i) = p_k - \frac{\frac{\partial G(p_k)}{\partial x_k}\left(\int_{x_{k+1}}^{x_k} v(q, t_i)dq + p_{k+1}x_{k+1} - p_k x_k\right)}{G(p_k) - G(p_{k-1})}$$

In our data, we observe unique bid quantities for only a subset of the prices on the price grid. In order to express the valuation as a function of the observed bidpoints, we must impose more structure on the data. In particular, we require that $G(p_j) = G(p_{j-1})$ at unobserved bidpoints j in order to derive an expression for valuation at the observed bidpoints. Intuitively, if the bidder submits unique bidpoints at, say p_5 and p_2 but not at p_3 and p_4, the bidder believes that between p_5 and p_2, the rivals will not submit bids that will cause residual supply to be between x_5 and x_2. Further details and intuition are provided in the Appendix. Note that in the other case of the discriminatory auction, this restriction is sufficient, but the first-order conditions could be expressed even if the condition does not hold (and instead there were costs to adding bidpoints).

Letting k^m index the observed bidpoints, the first-order condition can be expressed as:

$$(2) \qquad v(x_{k^m}, t_i) = p_{k^m} - \frac{\frac{\partial G(p_{k^m})}{\partial x_{k^m}} \left(\int_{x_{k^{m+1}}}^{x_{k^m}} v(q, t_i)dq + p_{k^{m+1}} x_{k^{m+1}} - p_{k^m} x_{k^m} \right)}{G(p_{k^m}) - G(p_{k^{m-1}})}$$

If we assume that the marginal valuation is a step function as in the discriminatory case, the integral in the above equation becomes: $v(x_{k^m})(x_{k^m} - x_{k^{m+1}})$. This yields a set of linear equations that we can solve for $v(x_{k^m}, t_i)$.

To provide intuition behind these equilibrium conditions (1) and (2), it would help to analyze the first-order conditions when bidders choose smooth continuous functions rather than discrete step functions. In the continuous bid function formulation of Wilson's share auction model, firms submit bid schedules $y(p)$ and $x(p)$, and the market clearing price (p^c) can be defined as $y_i(p^c) = Q - \sum_{j \neq i}^{N} y_j(p^c)$. The first order conditions for the two auction formats are given by:[18]

Discriminatory auction:

$$(3) \qquad v(y_i(p), t_i) = p + \frac{H(p, y_i(p))}{H_p(p, y_i(p))}$$

Uniform-price auction:

$$(4) \qquad v(x_i(p), t_i) = p - \frac{G_x(p, x_i(p))}{G_p(p, x_i(p))} \cdot x(p)$$

First, consider the discriminatory auction where the bidder pays the price bid on each unit that is won. The true valuation is the bid price plus a bid shading term given by H/H_p. The denominator of the bid shading term is the density of the market clearing price. The numerator is the probability that the market-clearing price is less than p. For prices much higher than the expected market-clearing price, the numerator is close to one, so the markdown is large. Bidders significantly shade bids that are 'likely to be' inframarginal because lowering the bid price lowers the payment for that unit but does not significantly lower the probability of winning the unit. However, as the bid price becomes closer to the expected market clearing price, the shading term is smaller. For units more 'likely to be' marginal, the bid shading is smaller because a lower bid price is trading off the lower payment conditional on winning against the lower probability of winning.

[18] For the reader's reference, a derivation for the continuous case is in the supplementary section.

Next, consider the logic behind the first-order condition for the uniform-price auction (4) where the bidder pays the market-clearing price for all units won. Again, the true value is the bid price plus a bid shading term that is increasing in the number of units won, $x(p)$. The ratio in the bid shading factor is given by G_x/G_p. The denominator is again the density of the market-clearing price. The numerator is the effect of increasing the bid quantity (holding price constant) on the distribution of the market-clearing price. Increasing the quantity demanded will increase the expected market-clearing price and therefore decrease the probability that the market clearing price is less than a given p, so $G_x \leq = 0$. Intuitively, as long as a bid price has some probability of setting the market-clearing price, the bidder will shade her bid for that unit. Bidders shade their bids more for higher quantities (holding G_x/G_p constant) because the lower market-clearing price lowers the price paid on *all* inframarginal units. This first-order condition is analogous to a monopsony buyer who faces uncertain residual supply and 'withholds' demand in order to reduce the market price.

The intuition from the first-order conditions of the continuous bid function case carries over to the discrete formulation. If we compare the first-order conditions (1) and (3) for the discriminatory format, it is straightforward to see the similarity. The discrete first-order condition is analogous to the continuous one with an additional term that accounts for the marginal effect of bidding on expected payoffs for quantities between bidpoints. Similarly for uniform-price auctions, equation (2) is a discrete analog to (4).

It is worth noting that recent work by Kastl [2006] shows that estimating bidders' marginal valuations using a model assuming continuously differentiable bid functions may introduce some bias. In his model of equilibrium bidding in step functions (which differs from our formulation of step function bidding), a bidder may bid a price higher than his marginal valuation. The intuition is most easily understood for a price-taker where there is no bid shading factor; the bidder equates marginal value with the expectation of the market-clearing price, conditional on her bid. In a uniform-price auction with steps, there may be positive probability that the market clearing price is below the bid price, so bids in equilibrium may be above the marginal value. As a result, the *ex post* revenue generated by an auction with step functions could be higher than revenue from a multi-unit Vickrey auction in which bidders bid their true marginal valuation.

III(ii). *Empirical Strategy*

For each auction, we estimate each bidder's valuation function using the first-order conditions (1) and (2). We use the estimated valuations to construct counterfactual measures of efficiency and revenue if each auction had been conducted under alternative auction formats. Note that (1) and (2) express unobserved bidder valuations' entirely as functions of observed bids and the distribution of the market-clearing price ($H(\cdot)$ and $G(\cdot)$). If we estimate the distribution of market clearing price, then marginal valuations, $v(q(p,t_i),t_i)$, corresponding to each point on the bid function, $q(p,t_i)$, can be identified.

Hortaçsu [2002] proposes a resampling procedure to estimate $H(\cdot)$ and $G(\cdot)$. In order to motivate the procedure, consider the most straightforward (yet infeasible) approach. Suppose that the researcher knew the distribution of bidders' private signals and could compute the equilibrium mapping from signals to bids. The researcher would simulate N draws from the signal distribution, compute the equilibrium bid for each of the N bidders, and find the market-clearing price. By repeating this procedure a large number of times, the researcher could construct the distribution of market-clearing prices. Unfortunately, data are not available on the signal distribution and the literature does not provide closed-form solutions to equilibrium bids of multiunit discriminatory and uniform-price auctions.

However, we can estimate $H(\cdot)$ and $G(\cdot)$ by making assumptions about how the observed bids are generated. Under our model of equilibrium bidding, firms have a common prior on the distribution of independent private signals and an equilibrium mapping from those signals to the expected profit-maximizing bids. If firms are playing a pure strategy Bayesian-Nash equilibrium of the game, then the observed bids capture firms' beliefs about the distribution of the market-clearing price. Hortaçsu [2002] suggests a resampling procedure that allows us to recover estimates of $H(\cdot)$ and $G(\cdot)$ from observed bids. This approach is appealing because it neither requires us to know (or assume) the distribution of private values nor compute a closed-form representation of the equilibrium bidding strategy.[19]

The resampling procedure in Hortaçsu [2002] is given by:

1. Fix bidder i among the N_t bidders in auction t.
2. From the sample of N_t bid vectors in auction t, draw a random sample of (N_t -1) bid vectors with replacement where a probability (1/ N_t) is placed on each vector from the original sample.
3. Using bidder i's observed bid vector and the (N_t -1) resampled bid vectors, find the market-clearing price where aggregate demand equals total supply. This yields a resampled realization of the market-clearing price, conditional on bidder i's bid vector.
4. Repeat steps 1-3 for each bidder B (a large number) times.
5. Repeat steps 1-4 for each bidder i in auction t.

[19] This resampling approach is an extension of work by Elyakime, Laffont, Loisel and Vuong [1994] and Guerre, Perrigne and Vuong [2000]. This approach uses nonparametric techniques, but parametric approaches also are available. For example, see Donald and Paarsch [1993] and Laffont, Ossard and Vuong [1995].

This procedure yields a resampled distribution of market-clearing prices for each bidder in each auction, conditional on the bidder's observed bid vector. $H(p, \overrightarrow{y}_i)$is estimated by counting the fraction of draws when the resampled market-clearing price is less than any given p. Hortaçsu [2002], Proposition 1, part 2 derives conditions for the consistency of the resampling estimator using data from a single auction.[20]

Our analysis of the uniform-price auction requires an estimate of $G_x(p,x_i(p))$. To estimate this, we use the fact that the distribution of the market-clearing price can be represented as a function of the sum of N-1 rivals' bid quantities which are *i.i.d.* random variables from the perspective of bidder i. That is:

$$G(p_k, x_i(p)) = \Pr\{p_{k*} \leq p_k | x_i(p)\} = \Pr\{x_i(p_k) \leq Q - \sum_{j \neq i} x_j(p_k)\}$$

In words, given a price p_k, the probability that the market-clearing price is less than or equal to this price is the same as the probability that firm i's quantity demand bid is less than residual supply at p_k. Let residual supply at p, denoted $RS_i(p)$, represent the deterministic total supply net of the stochastic aggregate rival demand. Let $F(x_i (p_k))$ and $f(x_i (p_k))$ denote the cdf and pdf of $RS_i(p_k)$ *conditional on* $x_i (p_k)$, then $G(p_k,x_i (p_k)) = 1- F(x_i (p_k))$. Hence,

$$\begin{aligned} G_x(p_k, x_i(p)) &= \frac{\partial}{\partial x}(1 - \Pr\{RS_i(p_k) \leqslant x_i(p_k)\}) \\ &= \frac{\partial}{\partial x}\{1 - F(x_i(p_k))\} = -f(x_i(p_k)) \end{aligned}$$

Because the resampling estimator provides B resampled residual supply functions, we can estimate $f(x_i (p_k))$ at any price using kernel estimation.[21]

To summarize, we use the observed bids and estimates of $H(p,y_i)$ and $G(p,x_i)$ and derivatives of these distribution functions to estimate each bidder's valuation function in each auction.

[20] The estimator is consistent as N_t goes to infinity. In addition, a slight modification is required so the estimator is centered around the expectation of the market-clearing price rather than the observed market-clearing price. Hortaçsu [2002] conducts a Monte Carlo experiment and finds that the estimator described above without recentering performs quite well.

[21] We use a normal kernel function with bandwidth h set equal to the standard deviation of resampled residual supplies multiplied by $B^{-1/5}$. $\hat{G}_x(p_k, x(p)) = -\frac{1}{Bh}\sum_{i=1}^{B} K\left(\frac{RS_i(p_k) - x(p_k)}{h}\right)$. Alternatively, we calculate $G_x(.)$ numerically using $\frac{G(x,p_k) - G(x+\Delta x,p_k)}{\Delta x}$for 'small' Δx, but the results are very similar. We also use this approach to calculate $H_y()$ in the discriminatory auction first-order condition.

IV. RESULTS

IV(i). *Illustration of Bids and Estimated Valuations*

We illustrate our procedure by showing estimates of marginal valuation and the distribution of the market-clearing price for a bidder under each auction format. As an example of the discriminatory auctions, we choose the auction of November 15, 1999, in which 28 bidders (14 banks and 14 security houses) competed to purchase securities for what amounted to 1,184.9 billion KW (about $10 billion). A total of 108 bid points were submitted and 56 bids (52%) were successful. The cutoff yield was 8.37%. After converting the yield to price by setting this cutoff yield to be 10,000 KW, the range of bid prices was from 9,971.14 KW to 10,013.15 KW. Figure 2 depicts the aggregate bid functions.

We illustrate the estimation procedure for primary dealer #S11 who submitted 5 price-quantity pairs, as shown in Figure 3. This bidder's highest price bid was for a quantity comprising 6% of total supply and the lowest priced bid is for a quantity comprising 24% of supply. To estimate the marginal valuation function for PD#S11, we hold its bid constant. Then, we generate a random draw of 27 bid vectors from the sample of 28 bid vectors with replacement, giving equal probability of 1/28 to each bid vector in the original sample. This resampling is performed 5,000 times to generate $5{,}000 \times 27$ resampled bid vectors and 5,000 residual supply curves. By intersecting these 5,000 residual supply curves with PD #S11's bid function, we calculate 5,000 market clearing prices. From these 5,000 resampled market clearing prices, we construct the estimated distribution of market clearing price, $H(\cdot)$, by counting the frequency with which a given price level is above the market clearing price. Finally, we evaluate the first-order condition (1) with the observed bids and the estimated $H(\cdot)$ to estimate the marginal valuation at each bidpoint.[22]

Figure 3 illustrates the marginal density of the market clearing price and the recovered marginal valuations for each observed bid point. Recall that bid shading is increasing in $H(\cdot)$, the probability that the market-clearing price is less than the bid price, and decreasing in H_p, the density of the market-clearing price. The estimated value is significantly larger than the bid price for PD #S11's highest priced bid because the market-clearing price is unlikely to be that high. For lower priced bids, the estimated value is just above the valuation because there is a smaller probability that the price will be below that bid.

One complication is that we may be unable to recover the marginal value for the lowest bid price either because we assume $H(p_0) = 0$, for $p_0 < p_{k1}$ so that $H(p_{k1-1})$in the numerator of (1) becomes zero, or because at sufficiently

[22] For comparison, we also estimate marginal valuation using the continuous first-order condition, and show results in the supplementary section.

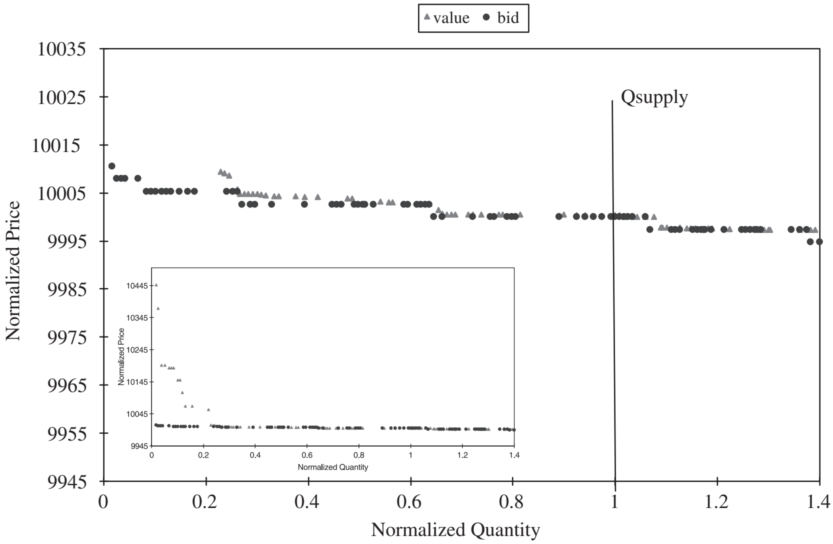

Figure 2
Discriminatory Auction of November 15, 1999. Aggregate Bids and Estimated Valuations. Note: The large graph focuses on the region surrounding the market-clearing price. The estimated valuation for the lowest quantity bids are not displayed. The inset graph contains all bids and estimated valuations

low prices $H(p_k) - H(p_{k-1})$ in the denominator is zero. This presents a practical problem for using estimated valuations in an aggregate analysis. We circumvent this problem by setting some restrictions on the valuation structure. One restriction is that the valuation function is weakly decreasing in quantity, and the other is that at lower price ranges below the expected market-clearing price, bidders are assumed not to shade prices below valuation. Therefore, if we fail to recover the true value for a winning bid, we assign $v_k = max(p_k, v_{k-1})$ at that point, where p_k is the bid price, and v_{k-1} is the recovered value at the next lowest bid price. If the missing value occurred at a losing bid, we assign its bid price as its true value. For example, in Figure 3, the valuation of the lowest bid price could not be estimated and was filled in with its bid price.[23] Table II shows the estimated valuations and jackknife standard errors. The estimates are much more precise for bids in

[23] Bids for which we could not recover the estimated valuation tend to occur at relatively low prices. We recover valuations for 67% of bids – 86% of bids above the market-clearing price and 55% of bids below the market-clearing price. Our rule for assigning valuation if it cannot be recovered can be seen as a lower bound and thus we may understate Vickrey revenues. However, our relatively high recovery rate for winning bids suggests that this bias is unlikely to be severe for purposes on comparing revenues.

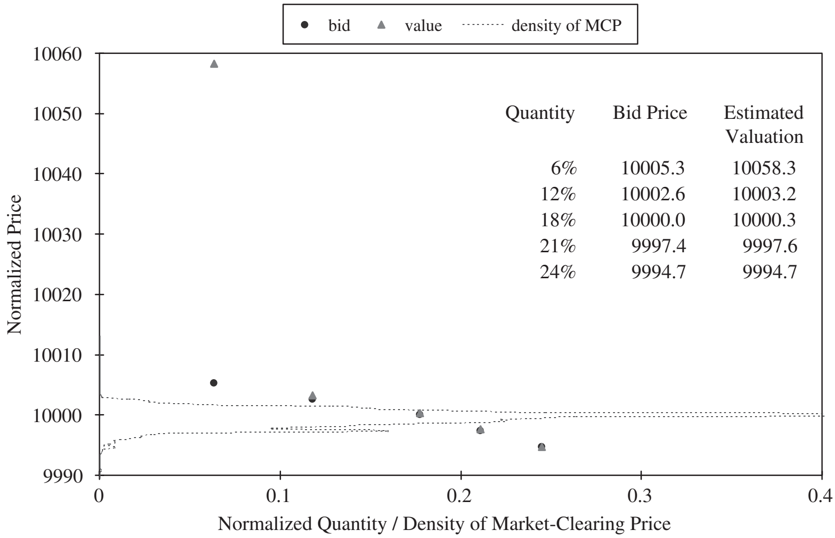

Figure 3
Discriminatory Auction of November 15, 1999. Bidder #S11

the price range where many bids occur than in ranges in which bids are infrequent.[24]

We repeat this analysis for every bidder in this auction. This gives us 28 bidders' point estimates of their marginal valuations rationalizing each price-quantity pair observed in this auction. We calculate the hypothetical competitive aggregate demand schedule by summing every individual's estimated marginal valuations schedule. Figure 2 displays the estimated aggregate marginal valuation function. The inset graph shows that bids are substantially flatter than valuations, as we expect for a discriminatory auction.

Figures 4 and 5 illustrate the bidder-level and aggregate analysis for a uniform-price auction estimated using equation (2). In the January 7, 2002, auction, 27 bidders (11 banks and 16 security houses) competed for what amounted to 1,200 billion Korean won (about $10 billion dollars). A total of 118 price-quantity bid pairs were submitted, and 37% of the bids were successful. The cutoff yield was 6.10% with a range of bid prices from 9,953.84 KW to 10,032.73 KW.

[24] The reason for the large standard error for the highest priced bid can be seen in the logic of the jackknife. Because the jackknife procedure removes a certain bidder's bid vector from the resampling set, the standard error can be large in areas where the number of observed bids is relatively scarce. Usually that area is away from the market-clearing price where H_p is relatively small.

TABLE II
EXAMPLE OF ESTIMATED MARGINAL VALUATIONS FOR REPRESENTATIVE BIDDERS UNDER EACH AUCTION FORMAT

Discriminatory Auction of November 15, 1999: Bidder #S11

Bid price	Quantity	Estimated marginal value	(SEjack)
10005.26	0.063	10058.26	707.70
10002.63	0.118	10003.16	1.11
10000.00	0.177	10000.25	0.36
9997.37	0.211	9997.60	0.46
9994.74	0.245	–	–

Uniform-Price Auction of January 7, 2002: Bidder #S2

Bid price	Quantity	Estimated marginal value	(SEjack)
10013.62	0.008	10013.62	–
10005.45	0.017	10005.49	0.06
10000.00	0.033	10000.57	0.38
9994.56	0.050	10000.66	133.89
9986.39	0.058	–	–

Note: The quantity is the ratio of the bidder's quantity over the total supply. As in Hortaçsu [2002], the standard errors are computed using the 'jackknife to bootstrap' method suggested by Efron [1992]. The 'jackknife to bootstrap' standard error for the estimate of the marginal valuation of bidder i for y_{ik} units of treasury bills, $s\tilde{e}_{jack}\{\hat{v}(y_{ik})\}$is given by:

$$s\tilde{e}_{jack}\{\hat{v}(y_{ik})\} = \left[\frac{n-1}{n}\left\{\sum_i (\bar{v}_{()} - \hat{v}_{(i)}(y_{ik}))^2\right\}\right]^{1/2}$$

where, $\hat{v}_{(i)}(y_{ik})$: the bootstrap estimate over a set of resamples that do not contain i's bid vector $\bar{v}_0 \equiv \sum_i \hat{v}_{(i)}(y_{ik})/n$

Bidder #S2 submitted 5 price-quantity pairs with the largest bid quantity comprising 6% of total supply. Our model of equilibrium bidding in a uniform-price auction says that the bid shading is increasing in the quantity purchased, $x_i\ (p)$, and in the effect of a single PD's bid on the market-clearing price G_x. As we see from Figure 5, the estimated bid shading factor is increasing in quantity for the first bid points. Table II contains the point estimates and jackknife standard errors of the estimated valuations.

Figure 4 illustrates the aggregate bid and estimated valuation functions. As compared to the discriminatory auction in Figure 2, the bid schedule is much steeper, reflecting the increased incentive to bid shade at larger quantities in the uniform-price format.

IV(ii). *Revenue Comparison*

The estimated valuation functions allow us to compare revenue in the actual auction to revenue under a counterfactual auction format. Unfortunately, we cannot make a direct revenue comparison because there do not exist closed-form solutions for multiunit auctions that allow us to use valuations

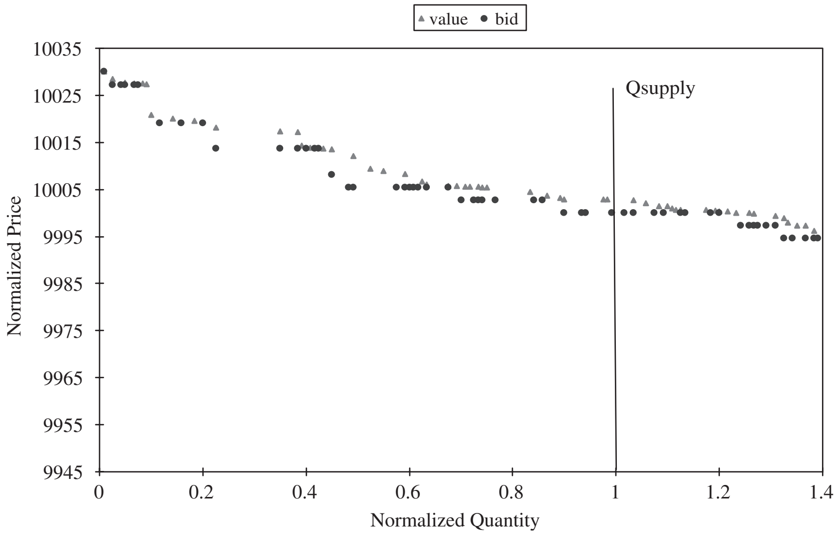

Figure 4
Uniform-Price Auction of January 7, 2002. Aggregate Bid and Estimated Valuations

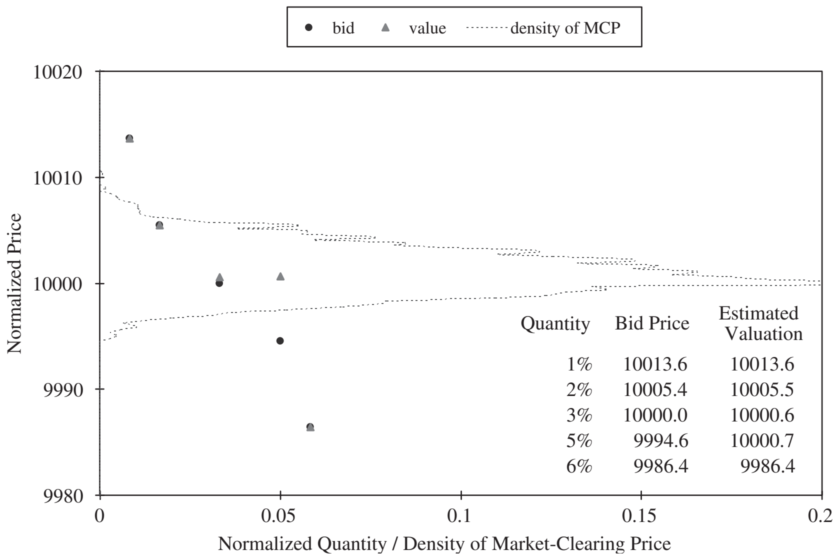

Figure 5
Uniform-Price Auction of January 7, 2002. Bidder #S2

to compute the equilibrium bidding under all alternative formats. Therefore, we compare the observed revenue under each format to a common benchmark for which we can compute equilibrium bidding. We use the multiunit Vickrey auction as our benchmark. In a Vickrey auction, it is a

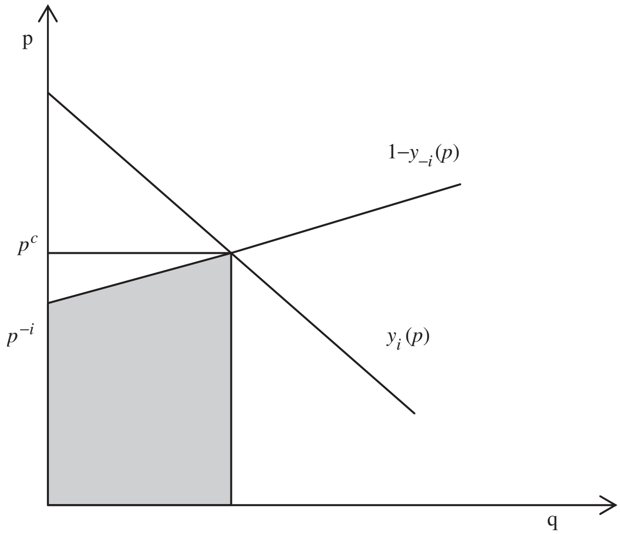

Figure 6
Payment Rule in Vickrey Auction.
Note: $y_{-i}(p)$ is the aggregate bid function by all bidders except bidder *i*.
The total supply, Q^{supply}, has been normalized to 1

dominant strategy equilibrium to bid the true valuation function. Because we estimate the marginal valuation function, we can calculate revenue under a Vickrey auction format and compare to observed revenue under the discriminatory and uniform-price formats.

In a Vickrey auction to sell *M* units of a good, the highest *M* bids win the good as in any standard auction. An individual bidder winning *k* units pays the sum of the k^{th} *highest rejected bids other than his own*. The Vickrey payment is depicted in Figure 6. The payment of bidder *i*, P_i, who wins $y_i(p^c)$ is given by:

$$P_i = y_i(p^c)p^c - \int_{p^{-i}}^{p^c} \{1 - y_{-i}(r)\}dr$$

where p^c: the market clearing price
p^{-i}: the market clearing price if bidder *i* had been absent from the auction
$y_{-i}(\cdot)$: the aggregate demand of all other bidders

Note that each bidder's payment is independent of his own bids conditional on the number of units won. Therefore, bidders do not have any incentive to shade bids, and consequently truth-telling is an equilibrium which yields an efficient allocation.

As a common benchmark, we compute an *upper bound* of revenue under the Vickrey auction as the market-clearing price under bidding the true valuation multiplied by the total supply, i.e. $p^{c*}Q^{supply}$ in Figure 6.[25] This upper bound for the Vickrey revenue is denoted R_v. We compare R_v to the *actual* revenue under either the discriminatory or uniform-price format, denoted R_a^d and R_a^u respectively. Note that R_v is always larger than actual revenue in the uniform-price format because R_v is equivalent to revenue in a perfectly competitive uniform-price auction, and we know that bidders shade bids in equilibrium.[26] However, R_v may be higher or lower than revenue under the discriminatory auction. Because R_a^u-R_v is necessarily negative, we can conclude that the discriminatory auction is revenue superior if R_a^d-R_v is significantly positive.

To illustrate the results, see the examples of aggregate bids and valuations for both types of auctions. In Figure 2 depicting a discriminatory auction, the actual revenue is given by the area under the aggregate bid schedule up to the total supply is $987.67 million U.S. dollars, while the upper bound of Vickrey revenue, is $987.44 million, representing a 0.023% *increase* in revenue relative to our benchmark. In Figure 4 depicting a uniform-price auction, the actual revenue, which is given by the rectangle formed by the intersection of the aggregate bid schedule and the total supply, is $1,000.00 million. The upper bound of Vickery revenue, calculated as the rectangle formed where aggregate valuation intersects supply, is $1000.28 million, representing a –0.03% revenue *loss* relative to our benchmark.

We perform the same procedure for all auctions, and Table III and IV summarize the results. In all discriminatory auctions, the actual revenues are higher than the benchmark upper bound of Vickrey revenues, with the revenue *gains* ranging from 0.02% to 0.10% and averaging 0.04%. In the uniform-price auctions, any bid shading causes actual revenue to be below our benchmark. The revenue *loss* ranges from nearly zero to 0.04% with an average of 0.027%.

We test whether these differences are statistically significant by accounting for the fact that our estimated marginal valuations are random variables. We construct bootstrapped standard errors. First, we construct 10,000 resamples of the pair of actual bids and estimated marginal valuations for each auction. In each resample there are N_t actual bid vectors and N_t marginal valuations vectors drawn randomly from the original set of bids

[25] We could compute true Vickrey revenue (i.e., the shaded region in Figure 6 summed over all bidders). This upper bound exceeds actual Vickrey revenue by $\sum_{i=1}^{N}\left(\int_{p^{-i}}^{p^c}\{1-y_{-i}(r)\}dr\right)$. However, this upper bound is easier to compute and is sufficient to make our revenue comparisons.

[26] The Kastl [2006] caveat regarding bidding above marginal valuation applies here. Kastl finds evidence of such behavior in 7 of the 28 auctions in his study.

TABLE III
REVENUE IN DISCRIMINATORY AUCTIONS VS. UPPER BOUND OF VICKREY

Date	Actual Revenue (R_a^d, \$mill)	Upper Bound of Vickrey Revenue (R_v, \$mill)	Revenue Difference (%) (= $[R_a^d–R_v]/R_v$)
9/13/1999	1001.26	1000.26	0.100
10/11/1999	1131.55	1131.17	0.033
11/15/1999	987.67	987.44	0.023
1/17/2000	637.33	637.08	0.039
2/14/2000	1023.63	1023.33	0.029
3/13/2000	553.40	553.25	0.027
4/10/2000	700.04	699.92	0.018
5/8/2000	635.11	635.00	0.017
6/12/2000	500.41	500.06	0.071
7/10/2000	488.58	488.33	0.051

Note: R_a^d is actual revenue in the discriminatory auction. R_v is calculated by measuring the rectangle which is formed by the intersection of the aggregate marginal valuation and the normalized total supply. A positive revenue difference (%) represents a revenue gain using the actual auction format relative to the benchmark.

TABLE IV
REVENUE IN UNIFORM-PRICE AUCTIONS VS. UPPER BOUND OF VICKREY

Date	Actual Revenue (R_a^u, \$mill)	Upper Bound of Vickrey Revenue (R_v, \$mill)	Revenue Difference (%) (= $[R_a^u–R_v]/R_v$)
8/14/2000	500.00	500.13	– 0.027
9/18/2000	750.00	750.01	– 0.002
10/9/2000	750.00	750.20	– 0.027
11/13/2000	791.67	791.88	– 0.027
1/8/2001	625.00	625.19	– 0.031
2/5/2001	641.67	641.84	– 0.028
3/12/2001	416.67	416.78	– 0.027
4/2/2001	666.67	666.91	– 0.037
5/7/2001	500.00	500.20	– 0.040
6/4/2001	333.33	333.42	– 0.027
7/2/2001	333.33	333.42	– 0.025
8/6/2001	583.33	583.49	– 0.028
9/3/2001	708.33	708.53	– 0.028
10/8/2001	741.67	741.83	– 0.022
11/7/2001	625.00	625.07	– 0.012
12/3/2001	925.00	925.26	– 0.028
1/7/2002	1000.00	1000.28	– 0.028
2/4/2002	333.33	333.42	– 0.027
3/4/2002	416.67	416.81	– 0.034
4/1/2002	475.00	475.13	– 0.027

Note: R_a^u is actual revenue in the uniform-price auction. R_v calculation is described in Table III. A negative revenue difference (%) represents a revenue loss using the actual format relative to the benchmark.

and the set of estimated marginal valuations vectors, respectively. With these resamples, we calculate the market clearing price and revenue for each pair in the resampling and obtain 10,000 differences of R_v and R_a. The mean of the resamples is our estimate of the *ex-ante* (expected) revenue difference, and the 2.5th and 97.5th percentiles yield our 95% confidence interval. Table V shows our results.

TABLE V
TEST FOR EXPECTED REVENUE DIFFERENCE (H_O:$R_A - R_V = 0$)

Format	Date	95% Confidence Interval for R_a–R_v†	Test Result
D	9/13/1999	[0.0001, 2.098]	Reject
D	10/11/1999	[0.232, 0.681]	Reject
D	11/15/1999	[0.162, 0.535]	Reject
D	1/17/2000	[0.134, 0.321]	Reject
D	2/14/2000	[0.174, 0.430]	Reject
D	3/13/2000	[0.093, 0.247]	Reject
D	4/10/2000	[0.018, 0.301]	Reject
D	5/8/2000	[0.088, 0.262]	Reject
D	6/12/2000	[0.108, 0.835]	Reject
D	7/10/2000	[0.108, 0.487]	Reject
UP	8/14/2000	[– 0.266, 0]	Not reject
UP	9/18/2000	[– 0.756, 0.987](– 0.791, 0)	Not reject (not reject)
UP	10/9/2000	[– 0.374, 0.069] (– 0.286, – 0.005)	Not reject (reject)
UP	11/13/2000	[– 0.425, – 0.023]	Reject
UP	1/8/2001	[– 0.363, 0.019] (– 0.303, 0)	Not reject (not reject)
UP	2/5/2001	[– 0.707, 0]	Not reject
UP	3/12/2001	[– 0.453, 0]	Not reject
UP	4/2/2001	[– 1.080, 0]	Not reject
UP	5/7/2001	[– 0.343, 0]	Not reject
UP	6/4/2001	[– 0.105, 0]	Not reject
UP	7/2/2001	[– 0.276, 0]	Not reject
UP	8/6/2001	[– 0.350, 0]	Not reject
UP	9/3/2001	[– 0.589, 0]	Not reject
UP	10/8/2001	[– 0.424, 0]	Not reject
UP	11/7/2001	[– 0.350, 0.142] (– 0.267, 0)	Not reject (not reject)
UP	12/3/2001	[– 0.763, – 0.253]	Reject
UP	1/7/2002	[– 0.541, 0]	Not reject
UP	2/4/2002	[– 0.146, 0]	Not reject
UP	3/4/2002	[– 0.254, 0]	Not reject
UP	4/1/2002	[– 0.379, 0]	Not reject

Note: R_a–R_v is the difference in revenue between the actual format used and our benchmark upper bound of Vickrey revenues.
†In four of the uniform-price auctions, the confidence intervals we compute include realizations where the actual revenue exceeds the Vickrey revenue (R_a^u–$R_v > 0$), which cannot occur if bid shading is non-positive. These results are driven by the part of the bid shading factor in the first-order condition (2) that accounts for quantities between the bidpoints which, in some cases, can take negative values. These terms are small and do not change the qualitative results. Therefore, we include in parentheses the confidence intervals computed without these terms.

For most of the uniform-price auctions, we cannot reject that the actual revenue is less than our benchmark. Actual revenues are bounded above by our benchmark, but the differences are not statistically significant. Because our benchmark revenue and actual revenues are identical under perfectly competitive bidding, these results suggest that the individual bidders do not possess a large amount of 'market power.' However, the discriminatory auctions yield higher expected revenue than our benchmark in all auctions. These results provide evidence that the discriminatory auctions in Korea yield higher expected revenue than uniform-price auctions.

Although we find statistically significant differences between the revenue of each format, the *economic* differences are arguably small. These results suggest that individual bidders do not have strong incentives to shade marginal bids. To see this, consider the source of incentives to bid shade under both formats from equations [3] and [4]. In uniform-price auctions, a bidder shades bids below valuations if the bidder can change the market-clearing price with her bid, i.e., $G_x < 0$ and the bidder has 'market power.' As the number of bidders grows large, the influence of a single bidder on the market-clearing price is smaller and bidders have incentives to bid their true valuation. Similarly, in a discriminatory auction, bidders have strong incentives to shade inframarginal bids but low incentives to shade marginal bids. Swinkels [1999] shows that as the number of bidders goes to infinity, bidders behave as price-takers at the market-clearing price. Intuitively, if there are enough firms so that there is no uncertainty about the market-clearing price, each bidder acts as a price-taker at that price. In the limit, the discriminatory auction is efficient. Consequently, both formats yield similar bid schedules and revenue in auctions with a large number of firms.

An explanation for the small percentage differences in revenue under each format is that the primary dealers in Korea operate in a fairly competitive market. Under either format, the bidders have little incentive to shade their marginal bids if they are essentially price-takers on a very elastic residual supply. A proxy for competitiveness is the *ex-post* residual supply elasticity at the market-clearing price. We calculate the residual supply elasticity at the market-clearing price of the largest bidder in each auction. Under auctions of both formats, the residual supply is very elastic. In the ten discriminatory auctions, the elasticity ranges from 334 to 1,355 with a mean elasticity of 919. The elasticities in the uniform-price auctions are smaller but still very elastic averaging 512. This suggests that bidders have low incentives to exercise monopsony power. These large residual supply elasticities help explain *statistically* different revenues that represent only a very small difference in *absolute* revenue. If the bidders are in equilibria of the discriminatory and uniform-price auctions that are close to the competitive equilibrium, then the auction formats may yield statistically different expected revenue that do not significantly differ in percentage terms.

It is noteworthy that Hortaçsu finds that the Turkish discriminatory price auction also yields higher revenue than the Vickrey benchmark. However, the difference in percentage terms is substantially larger than we find in Korea. Hortaçsu finds that *ex-post* revenues are 3.8% higher than the 'best-case' uniform price auction where bidders bid their true values (and this represents an upper bound for Vickrey revenues).

Finally, we should note that our revenue ranking is unlikely to be affected by the private values assumption. In the discriminatory auction, our estimates of the benchmark Vickrey revenues are likely to be biased upwards if the actual value structure has a common value component, as we describe below. Because we find actual discriminatory revenues to be higher than our estimates of Vickrey revenues, we can still conclude that actual discriminatory revenues are greater than Vickrey revenues. For a full derivation of the sign of the bias, see Hortaçsu [2002], Proposition 2. Briefly, it can be shown that optimal bidding in a multi-unit discriminatory auction with common values involves shading bids to adjust for a winners' curse. This winners' curse adjustment is exactly the difference between (a) the first-order condition (3) used to estimate the valuation (and equilibrium Vickrey bids) under private values, and (b) the equilibrium Vickrey bids under affiliated values. Thus, using the first-order condition under private values will overstate Vickrey revenues if values are actually affiliated. Unfortunately, it is difficult to quantify the sign of any bias in the uniform-price auction.

IV(iii). *Efficiency Comparison*

We can compute the efficiency of each format by measuring whether the actual format allocates the treasury bills to the bidders with highest valuations. Unlike previous research that analyzes only one format, we can compare the efficiency of discriminatory and uniform-price auctions because we observe both in Korea. It is worth emphasizing that one must observe actual bidding under both formats to compare the *efficiency* properties, while bidding data under a single format may suffice to compare the *revenue* properties. The reason is straightforward. It is possible to compare the revenue by observing bidding under just the discriminatory format because the Vickrey auction benchmark is an upper bound of uniform-price revenue and is possibly smaller than revenues under the actual discriminatory auction (as we find above).[27] However, both the uniform-price and discriminatory formats are in general inefficient relative to the Vickrey auction, so a bounding argument cannot be used for efficiency. As a result, the Korean Treasury's change in auction format provides a unique opportunity to study the efficiency properties of the two auction formats.

We measure efficiency loss as the difference in total surplus under the efficient allocation (e.g., a Vickrey auction) and the total surplus achieved under the allocation generated by the actual auction. Because we have estimated the marginal valuation for each bidder, it is straightforward to calculate surplus under each scenario.

Results are reported in Table VI. The efficiency loss is relatively small under both formats, which appears to reflect the fact that both mechanisms are asymptotically efficient as the number of firms grows large. The efficiency loss in uniform-price auction averages only 0.042%, but is still over 20 times larger than the average inefficiency in discriminatory auctions of 0.002%. If we analyze efficiency auction by auction, we see that there is

[27] This bounding approach is not applicable if the researcher only observes data from a uniform-price auction because the discriminatory auction revenues can be above or below the Vickrey revenues.

TABLE VI
EFFICIENCY LOSSES UNDER EACH AUCTION FORMAT

		Total Surplus (mill $US)		
Format	Date	Efficient Allocation	Actual Allocation	% Efficiency Loss
D	9/13/1999	1029.365	1029.299	0.006
D	10/11/1999	1132.345	1132.344	0.000
D	11/15/1999	990.443	990.439	0.000
D	1/17/2000	637.686	637.685	0.000
D	2/14/2000	1026.273	1026.273	0.000
D	3/13/2000	553.910	553.910	0.000
D	4/10/2000	704.670	704.670	0.000
D	5/8/2000	640.417	640.417	0.000
D	6/12/2000	533.415	533.396	0.004
D	7/10/2000	493.887	493.844	0.009
D	Mean	774.241	774.228	0.002
UP	8/14/2000	500.583	500.491	0.018
UP	9/18/2000	750.552	750.539	0.002
UP	10/9/2000	750.679	750.396	0.038
UP	11/13/2000	792.418	791.973	0.056
UP	1/8/2001	626.068	625.581	0.078
UP	2/5/2001	643.132	642.972	0.025
UP	3/12/2001	417.423	417.265	0.038
UP	4/2/2001	669.816	667.727	0.312
UP	5/7/2001	500.673	500.584	0.018
UP	6/4/2001	333.741	333.657	0.025
UP	7/2/2001	333.701	333.696	0.001
UP	8/6/2001	584.064	583.942	0.021
UP	9/3/2001	709.402	709.216	0.026
UP	10/8/2001	742.580	742.431	0.020
UP	11/7/2001	625.715	625.644	0.011
UP	12/3/2001	926.366	925.814	0.060
UP	1/7/2002	1001.124	1001.032	0.009
UP	2/4/2002	333.611	333.564	0.014
UP	3/4/2002	417.172	417.042	0.031
UP	4/1/2002	475.492	475.330	0.034
UP	Mean	606.716	606.445	0.042

Notes:
Surplus under *efficient* allocation is the total surplus if the K units sold go to bidders with the highest K marginal valuations. Surplus under *actual* allocation is the total surplus if the K units sold go to bidders with the highest K marginal bids.

some variation across auctions of the same format. However, the least efficient discriminatory auction is more efficient than all but three of the uniform-price auctions.

Although the efficiency losses are small under each format, we can nevertheless address the relative efficiency. In general, there is no efficiency ranking of the two formats in a multiunit setting (Ausubel and Cramton [2002]). The efficiency depends upon the structure of demand and the level of asymmetry across bidders. Inefficiency arises in the discriminatory auction because of differential incentives to bid shade across units. Theory does not predict whether this inefficiency is more likely to be generated by large or small firms.

In the uniform-price auction, inefficiency results from demand-reduction in order to lower the price paid on inframarginal units. For example, large bidders with more 'market power' have greater incentives to reduce demand. One might conjecture that this demand reduction causes smaller bidders to win units despite the fact that they have lower valuation. This conjecture is borne out in the data. We classify each bidder as 'large' or 'small' based upon the total bid and total winning bid quantities. For each of the uniform-price auctions, we calculate the number of each bidder's bidpoints that did not 'win' in the actual auction but would have won in a Vickrey auction. We find that 70% of these bids are submitted by large firms. This represents a disproportionately large share of bids by large bidders – large firms submit only 55% of total bids. This suggests that the uniform-price auction shifts the allocation away from high value large bidders towards lower value small bidders.

IV(iv). *Extending the Model to Allow for Asymmetric Primary Dealers*

In this section, we extend the model to allow for primary dealers to be *ex ante* asymmetric by size. Our conclusions are qualitatively very similar, however the magnitude of some of the revenue and efficiency differences do change. We divide bidders into two groups and assume the signal distribution to be equal within group and different across group. In the Korean market, we can expect two possible dimensions of asymmetries: between banks and security firms, and between large firms and small firms[28]. The bank-security firm classification is plausibly a source of asymmetry because these two groups differ in funding capability and the type of activity in the TB market.[29] On the other hand, the large-small classification focuses on asymmetries in ability to acquire information. Large bidders may have more advanced business networks which provide more information to forecast auction outcomes. We analyze summary statistics and find greater differences in bids along the large-small dimension.

Suppose that asymmetry is introduced between $N1$ large firms and $N2$ small firms. If we assume that the members of each group draw their signals independently from the distribution $F_1(\cdot)$ and $F_2(\cdot)$ respectively, and the two distributions are independent of each other, then we can interpret the residual supply for a large firm as the sum of total N-1 random variables

[28] The amount of assets, capital or revenue may be possible criteria for grouping, but these may not indicate how actively the bidders play in the Treasury market. Thus, to classify the size of the bidders, we use the total bid amounts and the total winning amounts by each bidder. In order to maintain consistency of the grouping of PDs, we exclude some firms that show a long discontinuity in participation; as a result, in three of the uniform-price auctions in 2002, we exclude a firm that actually bid.

[29] Banks typically have superior funding capability and are believed to participate in the auction to hold TBs; security houses are thought to purchase TBs for customers and to trade in the secondary market.

which are composed of *N1-1* random variables drawn from $F_1(\cdot)$ and *N2* drawn from $F_2(\cdot)$ independently. The definition of $H(\cdot)$ and the form of the first-order condition that we derived under the symmetric assumption does not change.[30]

Table VII compares results under the assumption of either symmetric or asymmetric signal distributions. The results do not change our qualitative conclusions that the discriminatory auction yields more revenue and a more efficient allocation, although the differences are still economically small. In terms of revenue, the discriminatory auction averages 0.041% more revenue than the Vickrey benchmark under both symmetric and asymmetric bidders. The uniform-price auction under symmetric and asymmetric bidders yields 0.027% and 0.025% less revenue then the Vickrey benchmark, respectively. In terms of efficiency, the results are again similar, however, losses for several of the auctions change quite a bit under asymmetric bidders. On average, the efficiency losses in discriminatory auctions under symmetry (asymmetry) are 0.002% (0.003%), while the losses in the uniform-price auctions under symmetry (asymmetry) are 0.042% (0.060%).

The differences in our estimated efficiency losses under symmetry and asymmetry highlight the contribution of large and small firms to efficiency losses. Recall that in section IV(iii) we show evidence that the uniform-price auction shifts the allocation away from large bidders to lower value small bidders. The results in Table VII are consistent with this phenomenon. Theory suggests that the large-small asymmetry is most likely to affect our results for the uniform-price auction; large bidders exhibit more bid shading (for large quantities) than small firms in the uniform-price auction, but bid shading is not necessarily related to firm size in the discriminatory auctions. When we allow for large and small firms to have different signal distributions, the estimated efficiency losses change a moderate amount for the uniform-price auction but very little for the discriminatory auction.

IV(v). *Other Robustness Results*

In this section, we test for the robustness of our results to the assumptions of common knowledge of the number of bidders and independence of private values.

[30] The validity of the resampling method is maintained with the asymmetries because the observed $q(p)$ is assumed to be the outcome of optimal strategic behavior even though the underlying mapping now becomes $(t_i, F_1(t), F_2(t)) \rightarrow q(p,t_i)$ instead of $(t_i, F(t)) \rightarrow q(p,t_i)$. However, we need to modify the resampling procedure slightly according to which group a bidder belongs. When resampling bid vectors, if *i* belongs to *N1* group, draw *N1-1* bid vectors from *N1* bid vectors by giving the same probability $1/N1$, and draw *N2* bid vectors from *N2* bid vectors by giving the same probability $1/N2$. With these resampled bid vectors ($N1\text{-}1 + N2$), construct the residual supply faced by bidder *i* and intersect *i*'s actual bid schedule to find market clearing price. Other steps are the same as the symmetric case.

TABLE VII
TESTING ROBUSTNESS TO ASYMMETRIC BIDDERS

Format	Date	% Efficiency Loss		% Revenue Loss	
		Symmetric	Asymmetric	Symmetric	Asymmetric
D	9/13/1999	0.006	0.006	0.100	0.117
D	10/11/1999	0.000	0.000	0.033	0.033
D	11/15/1999	0.000	0.000	0.023	0.024
D	1/17/2000	0.000	0.000	0.039	0.039
D	2/14/2000	0.000	0.000	0.029	0.029
D	3/13/2000	0.000	0.000	0.027	0.027
D	4/10/2000	0.000	0.000	0.018	0.018
D	5/8/2000	0.000	0.000	0.017	0.018
D	6/12/2000	0.004	0.003	0.071	0.080
D	7/10/2000	0.009	0.025	0.051	0.025
D	Mean	0.002	0.003	0.041	0.041
UP	8/14/2000	0.018	0.031	– 0.027	– 0.027
UP	9/18/2000	0.002	0.002	– 0.002	– 0.008
UP	10/9/2000	0.038	0.049	– 0.027	– 0.027
UP	11/13/2000	0.056	0.102	– 0.027	– 0.014
UP	1/8/2001	0.078	0.069	– 0.031	– 0.028
UP	2/5/2001	0.025	0.023	– 0.028	– 0.028
UP	3/12/2001	0.038	0.137	– 0.027	– 0.027
UP	4/2/2001	0.312	0.038	– 0.037	– 0.038
UP	5/7/2001	0.018	0.051	– 0.040	– 0.043
UP	6/4/2001	0.025	0.030	– 0.027	– 0.027
UP	7/2/2001	0.001	0.002	– 0.025	– 0.027
UP	8/6/2001	0.021	0.019	– 0.028	– 0.028
UP	9/3/2001	0.026	0.062	– 0.028	– 0.008
UP	10/8/2001	0.020	0.079	– 0.022	– 0.028
UP	11/7/2001	0.011	0.008	– 0.012	– 0.013
UP	12/3/2001	0.060	0.090	– 0.028	– 0.027
UP	1/7/2002	0.009	0.018	– 0.028	– 0.027
UP	2/4/2002	0.014	0.303	– 0.027	– 0.015
UP	3/4/2002	0.031	0.020	– 0.034	– 0.046
UP	4/1/2002	0.034	0.057	– 0.027	– 0.014
UP	Mean	0.042	0.060	– 0.027	– 0.025

Note: The methodology for these computations is described in section IV(iv).
'Symmetric' are results under the assumption that private values are independent draws from the same signal distribution.
'Asymmetric' are results under the assumption that private values are independent draws from one signal distribution for large firms and another for small firms. Bidders who do not repeatedly participate in auctions are excluded. Therefore, the 'symmetric' results differ slightly from those earlier but are included here for comparison purposes.

The model in section III assumes that the number of bidders in each auction is common knowledge. However, there is some variation is PD participation across the auctions. We modify our assumption and allow for the number of bidders to be uncertain. We assume that the number of *potential* bidders is common knowledge (recall that the list of registered PDs is public information) but the number of actual bidders is stochastic. We modify our resampling procedure to sample based on the number of potential bidders. For example, suppose there is an auction with 30 registered PDs, but only 28 bidders actually participate. We resample with $N_t = 30$, and anytime we draw one of the two bidders who did not

TABLE VIII
TESTING ROBUSTNESS TO UNCERTAINTY IN NUMBER OF BIDDERS

		Revenue Difference(%) (= [Ra-Rv]/Rv)		% Efficiency Loss	
Format	Date	Method #1	Method #2	Method #1	Method #2
D	9/13/1999	0.100	0.115	0.006	0.007
D	10/11/1999	0.033	0.033	0.000	0.000
D	11/15/1999	0.023	0.024	0.000	0.001
D	1/17/2000	0.039	0.039	0.000	0.000
D	2/14/2000	0.029	0.029	0.000	0.000
D	3/13/2000	0.027	0.027	0.000	0.000
D	4/10/2000	0.018	0.018	0.000	0.000
D	5/8/2000	0.017	0.017	0.000	0.000
D	6/12/2000	0.071	0.061	0.004	0.004
D	7/10/2000	0.051	0.025	0.009	0.078
UP	8/14/2000	– 0.027	– 0.027	0.018	0.029
UP	9/18/2000	– 0.002	– 0.002	0.002	0.001
UP	10/9/2000	– 0.027	– 0.027	0.038	0.006
UP	11/13/2000	– 0.027	– 0.019	0.056	1.389
UP	1/8/2001	– 0.031	– 0.043	0.078	0.063
UP	2/5/2001	– 0.028	– 0.028	0.025	0.015
UP	3/12/2001	– 0.027	– 0.027	0.038	0.014
UP	4/2/2001	– 0.037	– 0.037	0.312	0.211
UP	5/7/2001	– 0.040	– 0.054	0.018	0.067
UP	6/4/2001	– 0.027	– 0.027	0.025	0.006
UP	7/2/2001	– 0.025	– 0.023	0.001	0.001
UP	8/6/2001	– 0.028	– 0.055	0.021	0.105
UP	9/3/2001	– 0.028	– 0.031	0.026	0.020
UP	10/8/2001	– 0.022	– 0.028	0.020	0.064
UP	11/7/2001	– 0.012	– 0.028	0.011	0.082
UP	12/3/2001	– 0.028	– 0.028	0.060	0.183
UP	1/7/2002	– 0.028	– 0.028	0.009	0.009
UP	2/4/2002	– 0.027	– 0.024	0.014	0.030
UP	3/4/2002	– 0.034	– 0.055	0.031	0.064
UP	4/1/2002	– 0.027	– 0.027	0.034	0.053

Resampling method #1 is the method used throughout the paper, and is described in section III(ii). N_t is assumed to be common knowledge and equal to the number of *actual* bidders in each auction. Resampling method #2 is used to test for robustness, and is described in section IV(v). N_t is assumed to be the number of registered PDs, and when a non-participant in a particular auction is sampled, the bid vector is set to a quantity of 0.

participate, we include a bid schedule of zero quantity. We find revenue and efficiency results that are very similar. Table VIII compares the results under the two resampling procedures. The revenue differences are very similar under both resampling methods: the discriminatory auction averages 0.04% more revenue under both resampling approaches while the losses in the uniform-price auction are 0.002% and 0.009% respectively. As for the efficiency ranking, both methods yield similar results with the exception of two of the twenty uniform-price auctions when the modified resampling method suggests the uniform-price auction is even less efficient. Therefore, our results in terms of ranking and economic magnitudes do not appear to be affected by uncertainty in the number of bidders.

We also test our assumption that bidder signals are independent. We follow an approach similar to Bajari and Ye [2003] and test for independence in a reduced-form setting. Because bids are a function of signals, we test whether bids are independent after conditioning on a set of variables that are observable to all bidders at the time of the auction. We model bids with the following reduced-form:

$$BIDPRICE_{it} = \beta_0 + \beta_1 TSUPPLY_t + \beta_2 DISCRIM_t$$
$$+\beta_3 BANK_i + \beta_4 LARGE_i \beta_5 + UNCERTAIN_t + \beta_6 SRTREND_t + \varepsilon_{it}$$

where $BIDPRICE_{it}$ is a scalar measure of the probability-weighted average of each point in bidder i's bid vector, $TSUPPLY_t$ is the total supply sold in auction t, $DISCRIM_t$ is a dummy variable indicating if the format is discriminatory, $BANK_i$ is a dummy variable indicating if the bidder is a bank, $LARGE_i$ is a dummy variable indicating if the bidder is classified as large (see section IV(iv)), $UNCERTAIN_t$ is the standard deviation of market yields in the 10 days before the auction, and $SRTREND_t$ is the linearly fitted trend of resale market rates in the 30 days before the auction. After conditioning on factors affecting value that are publicly observable, we can interpret ε_{it} as a proxy for privately observed signals. Let the correlation coefficient between the vectors of residuals ε_{it} and ε_{jt} be denoted ρ_{ij}. Under the null hypothesis that signals are independent, H_o: $\rho_{ij} = 0$. Using the Fisher Z transformation where $Z = 1/2^{*}\ln((1+r)/(1-r))$with r the sample correlation coefficient, the test statistic $Z\sqrt{T-3}$ is approximately distributed standard normal under H_o. Among 525 pairs of bidders who compete against one another in at least 4 auctions, the null hypothesis is rejected for 100 pairs (or 19% of the pairs) at the 5% level. Under independence, only 5% of the pairs would reject the null. Banks tend to have more pairs with correlation between one another than do brokerages. Thus, our results above that assume independence should be viewed in the context that there is some evidence of signal correlation for at least some of the bidders. However, independence is not rejected for the majority of the combinations of bidders.

V. CONCLUSIONS

This paper finds that discriminatory auctions of Korean government securities lead to statistically higher expected revenue to the Treasury than uniform-price auctions. In addition, the discriminatory format better allocates the Treasury bills to the highest valuation banks and security houses. Previous empirical analyses using data from only a single format have been unable to make such efficiency comparisons. However, the economic differences are not large – the differences in both revenue and efficiency are much less than 1% between the two auction formats. We attribute this small difference to a highly competitive market that mitigates the strategic differences between the two formats. It is important to qualify this result by noting that models of bidding in uniform-price and discriminatory auctions can have multiple equilibria, and we only characterize the revenue and efficiency properties for equilibria observed in the Korean market during our sample period.

Our results suggest that discriminatory auctions in Korea are more efficient and yield higher revenue, *holding constant the set of bidders*. But it has been argued that uniform-price auctions encourage greater participation because the strategic information necessary to compete is smaller (Friedman [1960], Tenario [1993] and Ausubel and Cramton [2002]). In fact, one of the major reasons that the Korean Treasury switched from discriminatory to uniform-price was that many participants claimed that the switch would induce greater participation and more aggressive bidding. For example, consider a small bidder's information costs of participating in the auctions. A infinitesimally small bidder needs only to know her true valuation to calculate the equilibrium bid in a uniform-price auction, but needs to know both her own valuation and the distribution of rivals' bids to calculate the equilibrium bid in a discriminatory auction. A selling mechanism that encourages participation and makes bidding more competitive could have important efficiency implications to balance against the static efficiency implications considered in this paper. Future work could explore how this change in auction format changes the long-run number of participants in the Korean auctions.

APPENDIX
DERIVATION OF THE FIRST ORDER CONDITIONS IN DISCRETE CASE

1. *Discriminatory Auction*[31]

We model a risk neutral bidder who submits the bid vector $\vec{y}: \{y_o \geqslant y_1 \geqslant \cdots \geqslant y_{K+1}\}$ along the arbitrarily fine price grid $p_0 < p_1 < \cdots < p_{K+1}$ with elements separated by Δp. We make several assumptions about the boundaries of this arbitrarily large price grid. First, assume that the lowest point on the grid is below the support of the distribution of the market-clearing price, i.e. $H(p_k, y(p)) = 0$ at p_0 and any prices below p_0. Second, assume the bidder does not submit a bidpoint at p_0, i.e. that the bidder also demands y_1 at p_0, so that $y_0 = y_1$. Third, assume the bidder submits zero demand at the highest price on the price grid, p_{K+1}, i.e. $y_{K+1} = 0$, and at all higher prices.

The bidder maximizes expected profit which is:[32]

$$\sum_{k=1}^{K+1} [\Pr\{Mkt\, Clr\, P = p_k; \vec{y}\}] * \{payoff\ on\, bids \geqslant p_k\}$$
$$= \sum_{k=1}^{K+1} [H(p_k, \vec{y}) - H(p_{k-1}, \vec{y})] \times \sum_{j=k}^{K+1} \left(\int_{y_{j+1}}^{y_j} v(q, t_i) dq - p_j(y_j - y_{j+1}) \right)$$

[31] The derivation for the discriminatory auction follows Nautz [1995] and Hortaçsu [2002].

[32] By the first and second assumptions, the expected profit is a summation from $k = 1 \ldots K+1$. The first assumption that $H(p_0) = 0$ also allows us to solve the first-order condition for the lowest bid price $k = 1$, as we see below.

In the summation above, most of the products with $H(p_k)$ and all the products with $H(p_{k-1})$ in the next term of the summation will cancel. In addition, the third assumption implies that the partial sum for $K+1$ is zero. The Lagrangian can be written as:

$$L = \sum_{k=1}^{K} H(p_k, \vec{y})\left(\int_{y_{k+1}}^{y_k} v(q, t_i)dq - p_k(y_k - y_{k+1})\right) + \lambda_k(y_k - y_{k+1})$$

The first-order condition at each point p_k on the price grid from $k = 1 \ldots K$ is:

$$\begin{aligned}[\text{A1}] \quad & H(p_k)[v(y_k, t_i) - p_k] + \frac{\partial H(p_k)}{\partial y_k}\left(\int_{y_{k+1}}^{y_k} v(q, t_i)dq - p_k(y_k - y_{k+1})\right) + \lambda_k \\ & - H(p_{k-1})[v(y_k, t_i) - p_{k-1}] + \frac{\partial H(p_{k-1})}{\partial y_k}\left(\int_{y_k}^{y_{k-1}} v(q, t_i)dq - p_{k-1}(y_{k-1} - y_k)\right) \\ & - \lambda_{k-1} = 0\end{aligned}$$

Note that $\frac{\partial H(p_{k-1})}{\partial y_k} = 0$ so the next to last term disappears. To see this, recall that the market-clearing price p_{k^*} is defined as the lowest price on the price grid such that there is excess supply (or alternatively, the lowest price where bidder i's residual supply is greater than bidder i's bid quantity). A change in $y_i(p_k)$ can affect whether the market-clearing price is p_k or greater than p_k, but it will not affect whether the price is p_{k-1} or below.

If the bidder submits strictly increasing quantity bids at every point on the price grid, then the monotonicity constraints are not binding, $\lambda_k = 0 \forall k = 1 \ldots K$. By adding and subtracting $H(p_{k-1})p_k$ and re-arranging, we can solve the first-order condition:

$$v(y_k, t_i) = p_k + \frac{H(p_{k-1})[p_k - p_{k-1}]}{H(p_k) - H(p_{k-1})} - \frac{\frac{\partial H(p_k)}{\partial y_k}\left(\int_{y_{k+1}}^{y_k} v(q, t_i)dq - p_k(y_k - y_{k+1})\right)}{H(p_k) - H(p_{k-1})}$$

However, in our data the bidders submit unique quantity bids at only a subset of the prices in the price grid. If a bidder does not submit a unique quantity at p_k, then implicitly $y_k = y_{k+1}$. As we discuss in the main text, there are several possible interpretations of the fact that a bidder does not submit a bid at every possible price point. One possibility is that there is some (unmodeled) cost to adding a bidpoint, and that cost outweighs the expected benefits of 'fine tuning' the bid function. Kastl [2006] derives a model of bidding in step functions that explicitly models the cost of submitting bidpoints, and he estimates bounds on the marginal cost using bid data in the Czech treasury auctions. Another interpretation, suggested by Nautz [1995] and Hortaçsu [2002], is that the monotonicity constraint is binding at the unobserved bidpoints. A sufficient condition for the monotonicity constraint to bind is that $H(p_j) = H(p_{j-1})$ at unobserved bidpoint j, as shown by Nautz, and is easily verifiable using (A1). Intuitively, if bidding at a higher price p_j instead of p_{j-1} cannot increase the probability of winning, it is not profitable to raise the bid to p_j. However, the monotonicity constraint need not bind in order to express the first-order conditions in terms of only observed bidpoints, as we show below. (We will show that in the case of the uniform-price auction, we must impose more structure on the data than in the discriminatory auction).

We need to express the first-order conditions in terms of only the observed bidpoints. Suppose that we observe (p_2, y_2), (p_5, y_5) and (p_6, y_6) but do not observe (p_3, y_3), (p_4, y_4). Implicitly the bidder is bidding quantities $y_5 = y_4 = y_3$. If we add the first-order conditions for $k = 3,4,5$, then the Lagrange multipliers at the unobserved bidpoints (λ_3 and λ_4) will cancel in successive equations.[33] Recognizing that $y_5 = y_4 = y_3$, and then adding and subtracting $H(p_2)p_5$, we obtain:

$$\text{[A2]} \qquad v(y_5, t_i) = p_5 + \frac{H(p_2)[p_5 - p_2]}{H(p_5) - H(p_2)} - \frac{\frac{\partial H(p_5)}{\partial y_5}\left(\int_{y_6}^{y_5} v(q, t_i)dq - p_5(y_5 - y_6)\right)}{H(p_5) - H(p_2)}$$

Note that this is expressed entirely in terms of the observed bidpoints.

More generally, we can express valuation in terms of observed bidpoints. Let k^m index the observed bidpoints and j index unobserved bid prices between those observed bidpoints, $k^{m-1} + 1 \leqslant j < k^m$. By adding the first-order conditions across j from $k^{m-1} + 1 \leqslant j \leqslant k^m$, we obtain:

$$v(y_{k^m}, t_i) = p_{k^m} + \frac{H(p_{k^{m-1}})[p_{k^m} - p_{k^{m-1}}]}{H(p_{k^m}) - H(p_{k^{m-1}})} - \frac{\frac{\partial H(p_{k^m})}{\partial y_{k^m}}\left(\int_{y_{k^{m+1}}}^{y_{k^m}} v(q, t_i)dq - p_{k^m}(y_{k^m} - y_{k^{m+1}})\right)}{H(p_{k^m}) - H(p_{k^{m-1}})}$$

In order to calculate the integral involving $v(q,t_i)$, we need to know the functional form of the valuation function between bid quantities. We assume the marginal valuation function is a step function which assumes constant values of $v(y_k^m)$ on (y_k^{m+1}, y_k^m), as illustrated in Figure 1. The integral in the above equation becomes $v(y_{k^m})(y_{k^m} - y_{k^{m+1}})$.Thus, we have derived a set of linear equations that we can solve to estimate the marginal valuation step function.

2. *Uniform Price Auction*

Here we extend the discrete formulation from Nautz and Hortaçsu to the uniform-price auction. Because the change in auction format affects the bidders' optimal behavior resulting in a different distribution of the market-clearing price (*MCP*), we use a different notation for the bid function and distribution of MCP denoted by $x(p)$ and $G(p,x(p))$, respectively. In the uniform-price auction, we impose more structure on the data regarding the unobserved bidpoints than in the discriminatory auction. We provide intuition for the additional restrictions below.

As with the discriminatory case, assume that all bids are restricted to lie upon an arbitrarily fine grid of prices: $p_0 < p_1 < \cdots < p_{K+1}$. Again, we make several assumptions about the bids and probability of the market-clearing price at the boundary points. First, we assume that the lowest point on the grid is below the support of the distribution of the market-clearing price, i.e., $G(p_k,x(p)) = 0$ at p_0 and any prices below p_0. Second, assume the bidder does not submit a bidpoint at p_0, i.e. that the bidder also demands x_1 at p_0, so that $x_0 = x_1$. Third, assume the bidder submits zero demand at the highest price on the price grid, p_{K+1}, i.e. $x_{K+1} = 0$, and at all higher prices.

[33] Depending upon the interpretation of unobserved bidpoints, λ_3 and λ_4 may or not may not be zero. Of course, at the observed bid prices, $\lambda_k = 0$.

The expected payoff is given by:

$$\sum_{k=0}^{K+1}[\Pr\{MCP = p_k) * (\text{payoff if } MCP = p_k)]$$
$$= \sum_{k=0}^{K+1}[G(p_k, \vec{x}) - G(p_{k-1}, \vec{x})]\left(\int_0^{x_k} v(q, t_i)dq - p_k x_k\right)$$

Using the fact that

$$\int_0^{x_k} v(q, t_i)dq - p_k x_k = \int_0^{x_{k-1}} v(q, t_i)dq - p_{k-1}x_{k-1}$$
$$- \int_{x_k}^{x_{k-1}} v(q, t_i)dq - p_k x_k + p_{k-1}x_{k-1}$$

and incorporating the monotonicity constraints, we write the Lagrangian as:

$$L = \sum_{k=0}^{K+1}[G(p_k, \vec{x}) - G(p_{k-1}, \vec{x})]\left(\int_0^{x_{k-1}} v(q, t_i)dq - p_{k-1}x_{k-1}\right.$$
$$\left. - \int_{x_k}^{x_{k-1}} v(q, t_i)dq - p_k x_k + p_{k-1}x_{k-1}\right) + \lambda_k(x_k - x_{k+1})$$

By imposing the first and second assumptions about the boundary points, the partial sum for $k = 0$ disappears. In the remaining partial sums, several terms in each partial sum conveniently cancel with terms in the following partial sum. After imposing the first and third assumptions above, we obtain a summation from $k = 1 \ldots K$ (all interior points of the price grid):

$$L = \sum_{k=1}^{K} G(p_k, \vec{x})\left(\int_{x_{k+1}}^{x_k} v(q, t_i)dq + p_{k+1}x_{k+1} - p_k x_k\right) + \lambda_k(x_k - x_{k+1})$$

The first order condition at each point p_k on the price grid from $k = 1..K$ is:

$$\begin{aligned}[A3] \quad & G(p_k)[v(x_k, t_i) - p_k] + \frac{\partial G(p_k)}{\partial x_k}\left(\int_{x_{k+1}}^{x_k} v(q, t_i)dq + p_{k+1}x_{k+1} - p_k x_k\right) + \lambda_k \\ & - G(p_{k-1})[v(x_k, t_i) - p_k] + \frac{\partial G(p_{k-1})}{\partial x_k}\left(\int_{x_k}^{x_{k-1}} v(q, t_i)dq + p_k x_k - p_{k-1}x_{k-1}\right) \\ & - \lambda_{k-1} = 0\end{aligned}$$

Note that $\frac{\partial G(p_{k-1})}{\partial x_k} = 0$ for the same reason that $\frac{\partial H(p_{k-1})}{\partial y_k} = 0$ in the discriminatory case.

If a bidder submits a unique bid quantity at every point on the price grid, then the monotonicity constraints are not binding, i.e. $\lambda_k = 0 \forall k = 1 \ldots K$. Therefore, we can solve the first-order condition:

$$v(x_k, t_i) = p_k - \frac{\frac{\partial G(p_k)}{\partial x_k}\left(\int_{x_{k+1}}^{x_k} v(q, t_i)dq + p_{k+1}x_{k+1} - p_k x_k\right)}{G(p_k) - G(p_{k-1})}$$

However, in the empirical exercise we need to account for the fact that bidders submit unique quantity bids at only a subset of the prices in the price grid. Suppose that we observe (p_2, y_2), (p_5, y_5) and (p_6, y_6) but do not observe (p_3, y_3), (p_4, y_4). We need to express the first-order conditions in terms of only the observed bidpoints. Also, we need to provide an explanation for bidders choosing not to submit bids at some points.

Here we must impose more structure on the data than in the discriminatory case. We make an additional assumption (and give an economic interpretation of this assumption later in this section):

(*Assumption 1*) $G(p_2) = G(p_3) = G(p_4)$

The first equality of Assumption 1 implies that $\frac{\partial G(p_3)}{\partial x_3} = 0$ and the second equality implies $\frac{\partial G(p_4)}{\partial x_4} = 0$. Note that this assumption is not necessary in the discriminatory auction; in the discriminatory format, Assumption 1 need not hold and unobserved bidpoints could result from a cost of adding bidpoints.

If we add the first-order conditions [A3] for $k = 3,4,5$ and incorporate Assumption 1, then we obtain:

$$v(x_5, t_i) = p_5 - \frac{\frac{\partial G(p_5)}{\partial x_5}\left(\int_{x_6}^{x_5} v(q, t_i)dq + p_6 x_6 - p_5 x_5\right)}{G(p_5) - G(p_2)}$$

Note that this is expressed entirely in terms of the observed bidpoints.

More generally, we can express valuation in terms of observed bidpoints. Let k^m index the observed bidpoints and j index unobserved bid prices between those observed bidpoints, $k^{m-1} + 1 \leqslant j < k^m$. By adding the first-order conditions across j from $k^{m-1} + 1 \leqslant j \leqslant k^m$, we obtain:

$$v(x_{k^m}, t_i) = p_{k^m} - \frac{\frac{\partial G(p_{k^m})}{\partial x_{k^m}}\left(\int_{x_{k^{m+1}}}^{x_{k^m}} v(q, t_i)dq + p_{k^{m+1}} x_{k^{m+1}} - p_{k^m} x_{k^m}\right)}{G(p_{k^m}) - G(p_{k^{m-1}})}$$

If we assume that the marginal valuation is a step function as in the discriminatory case, the integral in the above equation becomes: $v(x_{k^m})(x_{k^m} - x_{k^{m+1}})$. This yields a set of linear equations that we can solve for $v(x_{k^m}, t_i)$.

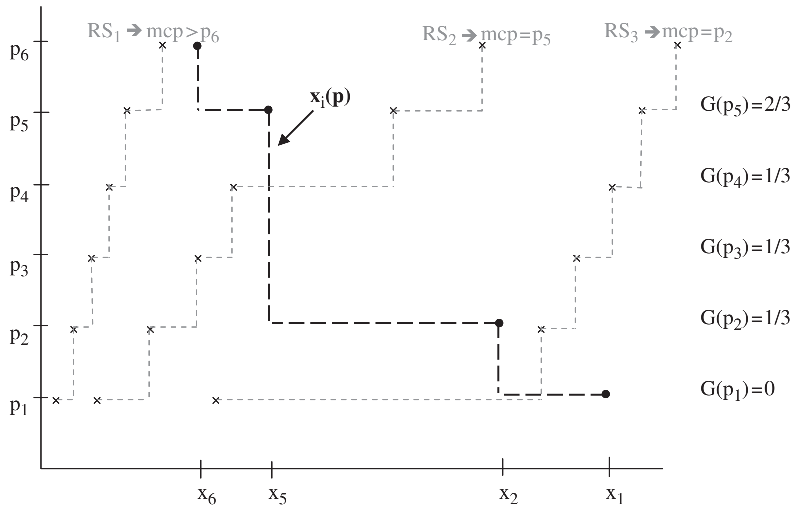

Figure
A1

Now, we provide intuition for Assumption 1 and the failure to submit bids at some price points. Consider Figure A1. Suppose that this bidder faces a Residual Supply function of either RS_1, RS_2, or RS_3 with equal probability of 1/3. Using the definition of the market-clearing price, we see that if RS_1 is realized, the market-clearing price is greater than p_6; if RS_2 is realized, the market-clearing price is p_5; and if RS_3 is realized, the market-clearing price is p_2. We can solve for $G(p_k)$ as shown in the figure. This $G(p_k)$ function satisfies Assumption 1 – $G(p_2) = G(p_3) = G(p_4)$, or more generally that at the unobserved bidpoints indexed by j, $k^{m-1} + 1 \leq j < k^m$, it must be the case that $G(p_{k^{m-1}}) = G(p_j)$.

One can see that $\frac{\partial G(p_j)}{\partial x_j} = 0$ at unobserved bidpoints in the context of Figure A1. If the bidder were to submit a unique bid quantity at either p_4 or p_3, then the quantity would have to be between x_5 and x_2 in order to satisfy the monotonicity constraints. There is not an x_4 (or x_3) satisfying $x_5 < x_4 < x_2$ that would change the market-clearing price under any of the possible realizations of Residual Supply. Therefore, at the unobserved bid prices, any bid quantity that satisfies the monotonicity constraints will not change $G(p_k)$ at that price, which is to say that $\frac{\partial G(p_3)}{\partial x_3} = 0 = \frac{\partial G(p_4)}{\partial x_4}$.

What's the interpretation of these restrictions from the bidder's perspective? The bidder believes that between p_5 and p_2, the rivals will not submit bids that will cause Residual Supply to be between x_5 and x_2. Of course, the rivals may submit some bid quantities at p_4 and/or p_3 (as is the case in the figure), but those bids cannot cause the Residual Supply to be between x_5 and x_2 at p_4 and/or p_3.

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